PROBLEM OF THE MONTH
SEPTEMBER 2015

Let f_n be the Fibonacci sequence. That is:

$$f_{n+2} = f_{n+1} + f_n \text{ with } f_1 = f_2 = 1.$$

Determine:

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{f_{2n+1}}\right).$$

Hint: Use a trig identity and the following identity which holds for the Fibonacci sequence: $f_{n+1}f_{n+2} - f_n f_{n+3} = (-1)^n$.

Solution:

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{f_{2n+1}}\right) = \frac{\pi}{4}.$$

First recall the trig identity:

$$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}.$$

Denoting $z = \tan(x)$, $w = \tan(y)$, and taking $\tan^{-1}$ of both sides of this equation gives:

$$\tan^{-1}(z) - \tan^{-1}(w) = \tan^{-1}\left(\frac{z-w}{1+zw}\right).$$

Now substitute: $z = \frac{1}{f_{2n}}$ and $w = \frac{1}{f_{2n+1}}$ to get:

$$\tan^{-1}\left(\frac{1}{f_{2n}}\right) - \tan^{-1}\left(\frac{1}{f_{2n+1}}\right) = \tan^{-1}\left(\frac{\frac{1}{f_{2n}} - \frac{1}{f_{2n+1}}}{1 + \frac{1}{f_{2n}}\frac{1}{f_{2n+1}}}\right) = \tan^{-1}\left(\frac{f_{2n+1} - f_{2n}}{f_{2n+1}f_{2n} + 1}\right).$$

Using the identity:

$$f_{2n+1}f_{2n} - f_{2n-1}f_{2n+2} = (-1)^{2n-1} = -1$$

and that

$$f_{2n+1} - f_{2n} = f_{2n-1}$$

we see that we get:

$$\tan^{-1}\left(\frac{1}{f_{2n}}\right) - \tan^{-1}\left(\frac{1}{f_{2n+1}}\right) = \tan^{-1}\left(\frac{f_{2n-1}}{f_{2n-1}f_{2n+2}}\right) = \tan^{-1}\left(\frac{1}{f_{2n+2}}\right).$$

Thus,

$$\tan^{-1}\left(\frac{1}{f_{2n+1}}\right) = \tan^{-1}\left(\frac{1}{f_{2n}}\right) - \tan^{-1}\left(\frac{1}{f_{2n+2}}\right)$$

Thus summing both sides we see that the terms on the right-hand side telescope and since $f_n \to \infty$ as $n \to \infty$ so that $\frac{1}{f_n} \to 0$ we get:

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{f_{2n+1}}\right) = \sum_{n=1}^{\infty}\left[\tan^{-1}\left(\frac{1}{f_{2n}}\right) - \tan^{-1}\left(\frac{1}{f_{2n+2}}\right)\right] = \tan^{-1}\left(\frac{1}{f_2}\right) - \tan^{-1}(0) =$$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$