SOLUTION FOR JANUARY 2016

A correct solution was turned in by

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Problem from December 2015

Show that the only solution of:

$$m^2 + n^2 + p^2 = 2mnp$$

where $m, n,$ and p are integers is $m = n = p = 0$.

Proof Notice that if $m = n = p = 0$ then the above equation is satisfied so now let us suppose (m, n, p) is a solution with m, n and p all nonzero and further let us assume that m, n and p do not have a common factor. It follows that since

$$m^2 + n^2 + p^2 = 2mnp \tag{1}$$

then at least one of m, n and p is even because the right-hand side of (1) is even and if m, n and p were all odd then m^2, n^2, and p^2 would all be odd and their sum would be odd so the left-hand side of (1) would be odd but the right-hand side is even and so we would obtain a contradiction.

So now let us assume without loss of generality that p is even. So then $p = 2q$ for some integer q and substituting into (1) and rewriting yields

$$m^2 + n^2 = 4mnq - 4q^2 = 4q(mn - q). \tag{2}$$

Thus the right-hand side of (2) is even so then either both m and n are even or both m and n are odd. Now m and n cannot both be even because if so then we would have that m, n and p are all even but we assumed that m, n and p have no common factor. Thus m and n must both be odd. Thus $m = 2m_1 + 1$ and $n = 2n_1 + 1$ for some integers m_1 and n_1. Substituting this into (2) and multiplying this out gives

$$4m_1^2 + 4m_1 + 4n_1^2 + 4n_1 + 2 = 4q(mn - q).$$

Dividing by 2 gives

$$2m_1^2 + 2m_1 + 2n_1^2 + 2n_1 + 1 = 2q(mn - q).$$

Now the left-hand side is odd and the right-hand side is even so we obtain a contradiction. So it must be that $m, n,$ or p is 0 and then it follows from (1) then the other two must also be 0. This completes the proof.