SOLUTION FOR FEBRUARY 2016 PROBLEM

Let a, b, p, x be real numbers with $0 < a \leq x \leq b$ and $p > 0$. Determine:

$$\min_{p} \max_{x \in [a,b]} \frac{|x-p|}{x}.$$

That is, first find the maximum of the function $\frac{|x-p|}{x}$ when $0 < a \leq x \leq b$. Let us call this number $M(p)$. Then find the minimum of the function $M(p)$ when $p > 0$.

SOLUTION:

$$\frac{b-a}{b+a}.$$

First we investigate the function $f(x) = \frac{|x-p|}{x}$ when $x > 0$ and $p > 0$. When $x > 0$ we may rewrite and obtain: $f(x) = |1 - \frac{p}{x}|$. We notice $f(p) = 0$, f is decreasing on $(0, p]$, and $\lim_{x \to 0^+} f(x) = +\infty$. Also note f is increasing on $[p, \infty)$ and that $\lim_{x \to \infty} f(x) = 1$. Let us denote $M(p) = \max_{x \in [a,b]} \frac{|x-p|}{x}$. We now investigate three cases.

Case 1: $0 < a \leq x \leq b < p$. In this case we see that f is decreasing so the maximum of f occurs at $x = a$ and so $M(p) = |1 - \frac{p}{a}| = \frac{p}{a} - 1$.

Case 2: $0 < p < a \leq x \leq b$. In this case we see that f is increasing so the maximum of f occurs at $x = b$ and so $M(p) = |1 - \frac{p}{b}| = 1 - \frac{p}{b}$.

Case 3: $0 < a \leq p \leq b$ and also $a \leq x \leq b$. In this case f is decreasing on $[a, p]$ and increasing on $[p, b]$. Thus $M(p) = \max\{\frac{p}{a} - 1, 1 - \frac{p}{b}\}$. It now turns out that $\frac{p}{a} - 1 \leq 1 - \frac{p}{b}$ precisely when $p \leq \frac{2ab}{a+b}$. Thus when $a \leq p \leq \frac{2ab}{a+b}$ then $M(p) = 1 - \frac{p}{b}$ and when $\frac{2ab}{a+b} < p \leq b$ then $M(p) = \frac{p}{a} - 1$.

Next it follows from Case 2 and Case 3 that when $0 < p \leq \frac{2ab}{a+b}$ then $M(p) = 1 - \frac{p}{b}$ and it follows from Case 1 and Case 3 that when $p > \frac{2ab}{a+b}$ then $M(p) = \frac{p}{a} - 1$. Therefore we see that $M(p)$ is decreasing when $0 < p \leq \frac{2ab}{a+b}$ and increasing when $p > \frac{2ab}{a+b}$. Thus the minimum of $M(p)$ occurs at $p = \frac{2ab}{a+b}$ and $M(\frac{2ab}{a+b}) = \frac{b-a}{b+a}$.