SOLUTION FOR MARCH 2016

Find all functions which satisfy:

$$f(x) + 2f\left(\frac{1}{1-x}\right) = x.$$

SOLUTION:

$$f(x) = \frac{1}{9}\left[x + 4 - \frac{4}{x} - \frac{2}{1-x}\right].$$

We begin with:

$$f(x) + 2f\left(\frac{1}{1-x}\right) = x. \tag{1}$$

Replacing x with $\frac{1}{1-x}$ gives:

$$f\left(\frac{1}{1-x}\right) + 2f\left(1 - \frac{1}{x}\right) = \frac{1}{1-x} \tag{2}$$

Now replacing x with $1 - \frac{1}{x}$ gives:

$$f\left(1 - \frac{1}{x}\right) + 2f(x) = 1 - \frac{1}{x}. \tag{3}$$

Multiplying equation (2) by -2 and adding to equation (1) gives:

$$f(x) - 4f\left(1 - \frac{1}{x}\right) = x - \frac{2}{1-x}. \tag{4}$$

Multiplying equation (3) by 4 and adding to equation (4) gives:

$$9f(x) = x + 4 - \frac{4}{x} - \frac{2}{1-x}.$$

Finally dividing by 9 yields the result.