

SOLUTION FOR APRIL 2016

Let $x > 0$ and $y > 0$. Let:

$$f(x, y) = \int_0^\infty \frac{1}{(1 + x^2 t^2)(1 + y^2 t^2)} dt.$$

Prove that:

$$f(x, y) = \frac{\pi}{2(x + y)}$$

and then calculate:

$$\int_0^1 \left(\int_0^1 f(x, y) dx \right) dy$$

and determine:

$$\int_0^\infty \frac{(\arctan t)^2}{t^2} dt.$$

SOLUTION: Using partial fractions we see that:

$$\begin{aligned} (x^2 - y^2)f(x, y) &= \int_0^\infty \frac{x^2 - y^2}{(1 + x^2 t^2)(1 + y^2 t^2)} dt = \int_0^\infty \frac{1}{1 + x^2 t^2} dt - \int_0^\infty \frac{1}{1 + y^2 t^2} dt \\ &= (x \tan^{-1}(xt) - y \tan^{-1}(yt)) \Big|_0^\infty = \frac{\pi}{2}(x - y). \end{aligned}$$

Thus for $x \neq y$ we see:

$$f(x, y) = \frac{\pi}{2(x + y)}.$$

We also notice that $f(x, y)$ is a continuous function and therefore it follows that:

$$f(x, y) = \frac{\pi}{2(x + y)} \text{ for all } x > 0, y > 0.$$

Next integrating in x we get:

$$\int_0^1 \frac{\pi}{2(x + y)} dx = \int_0^1 \int_0^\infty \frac{1}{(1 + x^2 t^2)(1 + y^2 t^2)} dt dx.$$

We now interchange the order of integration on the right-hand side to obtain:

$$\begin{aligned} \frac{\pi}{2} [\ln(1 + y) - \ln(y)] &= \int_0^\infty \int_0^1 \frac{1}{(1 + x^2 t^2)(1 + y^2 t^2)} dx dt \\ &= \int_0^\infty \frac{\tan^{-1}(xt) \Big|_0^1}{t} \left[\frac{1}{1 + y^2 t^2} \right] dt = \int_0^\infty \frac{\tan^{-1}(t)}{t} \frac{1}{1 + y^2 t^2} dt. \end{aligned}$$

Then integrating in y and changing the order of integration again gives:

$$\int_0^1 \frac{\pi}{2} [\ln(1 + y) - \ln(y)] dy = \int_0^1 \int_0^\infty \frac{\tan^{-1}(t)}{t} \frac{1}{1 + y^2 t^2} dt dy$$

$$= \int_0^\infty \int_0^1 \frac{\tan^{-1}(t)}{t} \frac{1}{1+y^2 t^2} dy dt = \int_0^\infty \frac{(\tan^{-1}(t))^2}{t^2} dt.$$

Finally working out the integral on the left we obtain:

$$\int_0^1 \frac{\pi}{2} [\ln(1+y) - \ln(y)] dy = \frac{\pi}{2} [(1+y) \ln(1+y) - y \ln(y)] \Big|_0^1 = \frac{\pi}{2} 2 \ln(2) = \pi \ln(2).$$

Thus,

$$\int_0^\infty \frac{(\tan^{-1}(t))^2}{t^2} dt = \pi \ln(2).$$