SOLUTION FOR SEPTEMBER 2016

Determine:

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} \left(n^2+i^2\right)^{1/n}.$$

Note: The notation $\prod_{i=1}^{2n} b_i$ means the product of these elements. That is $\prod_{i=1}^{2n} b_i = b_1 b_2 \cdots b_{2n-1} b_{2n}$.

SOLUTION:

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} \left(n^2+i^2\right)^{1/n} = 25e^{-4+2\tan^{-1}(2)}$$

First we write:

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} \left(n^2+i^2\right)^{1/n} = L$$

and then take logs of both sides and simplify using rules of logs to obtain:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{2n} \ln(n^2+i^2) - 4\ln(n) = \ln(L).$$

Next:

$$\ln(n^2+i^2) = \ln(n^2(1+\frac{i^2}{n^2})) = 2\ln(n) + \ln(1+\frac{i^2}{n^2})$$

and therefore:

$$\frac{1}{n}\sum_{i=1}^{2n} \ln(n^2+i^2) - 4\ln(n) = \frac{1}{n}\sum_{i=1}^{2n}\left(2\ln(n)+\ln(1+\frac{i^2}{n^2})\right) - 4\ln(n) = \frac{1}{n}\sum_{i=1}^{2n}\ln(1+\frac{i^2}{n^2}).$$

Next you might observe that this looks like a Riemann sum for the function $\ln(1+x^2)$ and in fact if we take a partition of $[0,2]$ by dividing this interval into $2n$ subintervals and if we use the right-hand endpoint of each subinterval as a selection point then we see that we obtain:

$$\int_0^2 \ln(1+x^2)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{2n} \ln(1+\frac{i^2}{n^2}).$$

Therefore we just need to calculate $\int_0^2 \ln(1+x^2)\,dx$. Integrating by parts gives:

$$\int_0^2 \ln(1+x^2)\,dx = x\ln(1+x^2)|_0^2 - \int_0^2 \frac{2x^2}{1+x^2}\,dx = 2\ln(5) - \int_0^2 \frac{2x^2}{1+x^2}\,dx$$

$$= 2\ln(5) - \int_0^2 \frac{2x^2+2}{1+x^2}\,dx + \int_0^2 \frac{2}{1+x^2}\,dx = 2\ln(5) - 4 + 2\tan^{-1}(x)|_0^2 = 2\ln(5) - 4 + 2\tan^{-1}(2).$$

Therefore:

$$\ln(L) = 2\ln(5) - 4 + 2\tan^{-1}(2)$$

and thus:

$$L = 25e^{-4+2\tan^{-1}(2)}.$$