SOLUTION FOR APRIL 2018

Determine:

$$\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right).$$

Hint: Note that:

$$z^n-1=(z-1)(z-e^{\frac{2\pi i}{n}})(z-e^{\frac{4\pi i}{n}})(z-e^{\frac{6\pi i}{n}})\cdots(z-e^{\frac{2(n-1)\pi i}{n}}).$$

SOLUTION:

$$\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right)=\frac{n}{2^{n-1}}.$$

Using the hint and factoring the left-hand side we obtain:

$$z^n-1=(z-1)(z^{n-1}+z^{n-2}+\cdots+z+1)$$

and so for $z\neq 1$ we obtain:

$$z^{n-1}+z^{n-2}+\cdots+z+1=(z-e^{\frac{2\pi i}{n}})(z-e^{\frac{4\pi i}{n}})(z-e^{\frac{6\pi i}{n}})\cdots(z-e^{\frac{2(n-1)\pi i}{n}}).$$

Now taking limits as $z\to 1$ we obtain:

$$n=(1-e^{\frac{2\pi i}{n}})(1-e^{\frac{4\pi i}{n}})(1-e^{\frac{6\pi i}{n}})\cdots(1-e^{\frac{2(n-1)\pi i}{n}}).$$

Multiplying both sides by $e^{\frac{-\pi i}{n}}$ gives:

$$ne^{\frac{-\pi i}{n}}=(e^{\frac{-\pi i}{n}}-e^{\frac{\pi i}{n}})(1-e^{\frac{4\pi i}{n}})(1-e^{\frac{6\pi i}{n}})\cdots(1-e^{\frac{2(n-1)\pi i}{n}})$$

and thus:

$$ne^{\frac{-\pi i}{n}}=(-2i)\sin\left(\frac{\pi}{n}\right)(1-e^{\frac{4\pi i}{n}})(1-e^{\frac{6\pi i}{n}})\cdots(1-e^{\frac{2(n-1)\pi i}{n}}).$$

Similarly multiplying by $e^{\frac{-2\pi i}{n}}$ gives:

$$ne^{\frac{-\pi i}{n}(1+2)}=ne^{\frac{-\pi i}{n}}e^{\frac{-2\pi i}{n}}=(-2i)^2\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)(1-e^{\frac{6\pi i}{n}})\cdots(1-e^{\frac{2(n-1)\pi i}{n}}).$$

Continuing in this way:

$$ne^{\frac{-\pi i}{n}(1+2+\cdots+(n-1))}=(-2i)^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right).$$

Since $1+2+\cdots+(n-1)=\frac{n(n-1)}{2}$ we obtain:

$$n(-i)^{n-1}=ne^{-\frac{(n-1)\pi i}{2}}=(-2i)^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right)$$

Therefore:

$$\frac{n}{2^{n-1}}=\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right).$$