SOLUTION FOR SEPTEMBER 2019

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SOLUTION: Let a and b be real with $b \neq 0$. Then:

$$\int_0^{2\pi} \frac{d\theta}{\cos^2(\theta) + 2a\cos(\theta)\sin(\theta) + (a^2+b^2)\sin^2(\theta)} = \frac{2\pi}{|b|}.$$

Proof: Since $\sin(\theta)$ and $\cos(\theta)$ are 2π periodic then we can integrate over any interval of length 2π and so the above integral can be rewritten as:

$$\int_{-\pi/2}^{3\pi/2} \frac{d\theta}{\cos^2(\theta) + 2a\cos(\theta)\sin(\theta) + (a^2+b^2)\sin^2(\theta)}.$$

Using that $\cos(\theta+\pi) = -\cos(\theta)$ and $\sin(\theta+\pi) = -\sin(\theta)$ we see then that the above integral becomes:

$$2\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos^2(\theta) + 2a\cos(\theta)\sin(\theta) + (a^2+b^2)\sin^2(\theta)}.$$

Rewriting gives:

$$2\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos^2(\theta) + 2a\cos(\theta)\sin(\theta) + (a^2+b^2)\sin^2(\theta)} = 2\int_{-\pi/2}^{\pi/2} \frac{\sec^2(\theta)d\theta}{1 + 2a\tan(\theta) + (a^2+b^2)\tan^2(\theta)}.$$

Now we let $x = \tan(\theta)$ so $dx = \sec^2(\theta)\,d\theta$ and thus:

$$2\int_{-\pi/2}^{\pi/2} \frac{\sec^2(\theta)d\theta}{1 + 2a\tan(\theta) + (a^2+b^2)\tan^2(\theta)} = 2\int_{-\infty}^{\infty} \frac{dx}{1+2ax+(a^2+b^2)x^2}.$$

Now let $x = \frac{y}{\sqrt{a^2+b^2}}$ so $dx = \frac{1}{\sqrt{a^2+b^2}}\,dy$ and thus:

$$2\int_{-\infty}^{\infty} \frac{dx}{1+2ax+(a^2+b^2)x^2} = \frac{2}{\sqrt{a^2+b^2}}\int_{-\infty}^{\infty} \frac{dx}{1+2cy+y^2}.$$

where $c = \frac{a}{\sqrt{a^2+b^2}}$. Note that since $b \neq 0$ then $-1 < c < 1$ thus $0 < 1 - c^2 \leq 1$.

Completing the square now gives:

$$\frac{2}{\sqrt{a^2+b^2}}\int_{-\infty}^{\infty} \frac{dy}{1+2cy+y^2} = \frac{2}{\sqrt{a^2+b^2}}\int_{-\infty}^{\infty} \frac{dy}{(y+c)^2+(1-c^2)} = \frac{2}{\sqrt{a^2+b^2}}\int_{-\infty}^{\infty} \frac{dz}{z^2+(1-c^2)}$$

$$= \frac{2}{\sqrt{a^2+b^2}}\frac{1}{\sqrt{1-c^2}}\tan^{-1}\left(\frac{z}{\sqrt{1-c^2}}\right)\Big|_{-\infty}^{\infty} = \frac{2\pi}{\sqrt{a^2+b^2}}\frac{1}{\sqrt{1-c^2}}.$$

Recalling $c = \frac{a}{\sqrt{a^2+b^2}}$ yields $\sqrt{1-c^2} = \frac{|b|}{\sqrt{a^2+b^2}}$ and thus finally:

$$\int_0^{2\pi} \frac{d\theta}{\cos^2(\theta) + 2a\cos(\theta)\sin(\theta) + (a^2+b^2)\sin^2(\theta)} = \frac{2\pi}{|b|}.$$