SOLUTION FOR SEPTEMBER 2021

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Determine:

$$\lim_{n\to\infty} n\sin(2\pi e n!).$$

Hint: $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$.

SOLUTION:

$$\lim_{n\to\infty} n\sin(2\pi e n!) = 2\pi.$$

Proof: Using the hint we see that:

$$n!e = \text{ integer } + \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \cdots = \text{ integer } + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots.$$

Thus:

$$2\pi n!e = 2\pi(\text{integer}) + \frac{2\pi}{n+1} + \frac{2\pi}{(n+1)(n+2)} + \cdots$$

and since $\sin(x)$ is 2π periodic we see that:

$$\sin(2\pi n!e) = \sin\left(\frac{2\pi}{n+1}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right)\right) = \sin\left(\frac{2\pi}{n+1}(1+\delta)\right)$$

where $0 < \delta = \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots < \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \cdots = \frac{1}{n+1}$.

Next we recall:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Thus for x sufficiently small we have:

$$1 - \epsilon < \frac{\sin(x)}{x} < 1 + \epsilon.$$

So for $x = \frac{2\pi}{n+1}(1+\delta)$ we have:

$$1 - \epsilon < \frac{\sin(\frac{2\pi}{n+1}(1+\delta))}{\frac{2\pi}{n+1}(1+\delta)} < 1 + \epsilon.$$

Thus:

$$n\sin(2\pi n!e) = 2\pi\left(\frac{\sin(\frac{2\pi}{n+1}(1+\delta))}{\frac{2\pi}{n}}\right) = 2\pi\left(\frac{\sin(\frac{2\pi}{n+1}(1+\delta))}{\frac{2\pi}{n+1}(1+\delta)}\right)\frac{n(1+\delta)}{n+1}.$$

Also since $0 < \delta < \frac{1}{n+1}$ we have:

$$(1 - \frac{1}{n+1}) < \frac{n(1+\delta)}{n+1} < 1.$$

Thus for n sufficiently large we have:

$$2\pi(1 - \frac{1}{n+1})(1-\epsilon) < n\sin(2\pi n!e) < 2\pi(1+\epsilon).$$

Thus:

$$\lim_{n\to\infty} n\sin(2\pi e n!) = 2\pi.$$

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