SOLUTION FOR MAY 2022

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Determine:

$$\sum_{n=2}^{\infty} \ln\left(1 + \frac{(-1)^n}{n}\right).$$

SOLUTION:

$$\sum_{n=2}^{\infty} \ln\left(1 + \frac{(-1)^n}{n}\right) = 0.$$

Proof: First recall by rules of logarithms that $\ln(1/x) = -\ln(x)$.

Next writing out the sum for the first several terms we see:

$$\ln\left(1 + \frac{1}{2}\right) = \ln\left(\frac{3}{2}\right)$$

$$\ln\left(1 + \frac{1}{2}\right) + \ln\left(1 - \frac{1}{3}\right) = \ln\left(\frac{3}{2}\right) + \ln\left(\frac{2}{3}\right) = 0$$

$$\ln\left(1 + \frac{1}{2}\right) + \ln\left(1 - \frac{1}{3}\right) + \ln\left(1 + \frac{1}{4}\right) = \ln\left(\frac{5}{4}\right)$$

$$\ln\left(1 + \frac{1}{2}\right) + \ln\left(1 - \frac{1}{3}\right) + \ln\left(1 + \frac{1}{4}\right) + \ln\left(1 - \frac{1}{5}\right) = \ln\left(\frac{5}{4}\right) + \ln\left(\frac{4}{5}\right) = 0$$

Similarly it follows that if n is odd then the sum of the first n terms of $\sum_{k=2}^{n} \ln\left(1 + \frac{(-1)^k}{k}\right) = 0$. Whereas if n is even then the sum of the first n terms of $\sum_{k=2}^{n} \ln\left(1 + \frac{(-1)^k}{k}\right) = \ln(1 + \frac{1}{n})$. Notice also that $\ln(1 + \frac{1}{n}) \to \ln(1 + 0) = \ln(1) = 0$ as $n \to \infty$.

Therefore, in both cases we see:

$$\sum_{k=2}^{\infty} \ln\left(1 + \frac{(-1)^k}{k}\right) = \lim_{n\to\infty} \sum_{k=2}^{n} \ln\left(1 + \frac{(-1)^k}{k}\right) = 0.$$