SOLUTION FOR OCTOBER 2023

Determine:

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx.$$

Solution:

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}.$$

Proof: First we observe that if we make the substitution $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$ (so that $-u^2\,du = dx$), and recall that $\ln(\frac{1}{u}) = -\ln(u)$ then we see that:

$$\int_1^\infty \frac{(\ln x)^2}{1+x^2}\,dx = \int_1^0 \frac{(\ln \frac{1}{u})^2}{1+\frac{1}{u^2}}(-u^2)\,du = \int_0^1 \frac{(\ln u)^2}{1+u^2}\,du = \int_0^1 \frac{(\ln x)^2}{1+x^2}\,dx.$$

Therefore we see that:

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx = \int_0^1 \frac{(\ln x)^2}{1+x^2}\,dx + \int_1^\infty \frac{(\ln x)^2}{1+x^2}\,dx = 2\int_0^1 \frac{(\ln x)^2}{1+x^2}\,dx.$$

Next for $-1 < x < 1$ we have:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^\infty x^n.$$

Replacing x by $-x^2$ gives for $-1 < x < 1$:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^\infty (-1)^n x^{2n}.$$

Thus we see:

$$\frac{(\ln x)^2}{1+x^2} = \sum_{n=0}^\infty (-1)^n x^{2n}(\ln x)^2.$$

Next we integrate by parts and obtain:

$$\int_0^1 x^{2n}\ln^2 x\,dx = \frac{x^{2n+1}}{2n+1}\ln^2 x\Big|_0^1 - \int_0^1 \frac{2x^{2n+1}}{2n+1}(\ln x)(\frac{1}{x})\,dx = 0 - \frac{2}{2n+1}\int_0^1 x^{2n}\ln x\,dx.$$

We integrate by parts again and get:

$$-\frac{2}{2n+1}\int_0^1 x^{2n}\ln x\,dx = -\frac{2}{(2n+1)^2}x^{2n+1}\ln x\Big|_0^1 + \frac{2}{(2n+1)^2}\int_0^1 x^{2n} = 0 + \frac{2}{(2n+1)^3}.$$

Therefore we see that:

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx = 2\int_0^1 \frac{(\ln x)^2}{1+x^2}\,dx = 4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3}.$$

Next it is a well-known fact that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$ and so we finally see that:

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2}\, dx = \frac{\pi^3}{8}.$$

□

Note: It is known that:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} = \frac{(-1)^k E_{2k}}{2^{2k+2}(2k)!} \pi^{2k+1}$$

where the E_{2k} are the *Eulerian numbers.* These numbers are integers and they come up in the Maclaurin series of $\sec(x)$.