SOLUTION FOR SEPTEMBER 2023

Let $0 < a < b$. Determine:

$$\sum_{n=1}^{\infty} \frac{e^{-na} - e^{-nb}}{n}.$$

Solution:

$$\ln\left(\frac{1-e^{-b}}{1-e^{-a}}\right).$$

Proof: We first consider

$$\sum_{n=1}^{N} e^{-nx} \text{ for } x > 0$$

and notice that the right-hand side is a finite geometric series with sum:

$$\sum_{n=1}^{N} e^{-nx} = \frac{e^{-x}(1-e^{-Nx})}{1-e^{-x}} \text{ for } x > 0.$$

Integrating on $[a, b]$ gives:

$$\sum_{n=1}^{N} \frac{e^{-na} - e^{-nb}}{n} = \int_a^b \frac{e^{-x}}{1-e^{-x}}\, dx - \int_a^b \frac{e^{-(N+1)x}}{1-e^{-x}}\, dx \tag{1}$$

$$= \ln\left(\frac{1-e^{-b}}{1-e^{-a}}\right) - \int_a^b \frac{e^{-(N+1)x}}{1-e^{-x}}\, dx. \tag{2}$$

Now since for $x > 0$ we have $0 < \sum_{n=1}^{\infty} \frac{e^{-nx}}{n} \leq \sum_{n=1}^{\infty} e^{-nx} = \frac{e^{-x}}{1-e^{-x}}$ it follows then that the left-hand side of (1) converges as $N \to \infty$ and thus from (1)-(2) we obtain:

$$\sum_{n=1}^{\infty} \frac{e^{-na} - e^{-nb}}{n} = \ln\left(\frac{1-e^{-b}}{1-e^{-a}}\right) - \lim_{N\to\infty} \int_a^b \frac{e^{-(N+1)x}}{1-e^{-x}}\, dx.$$

Finally we have:

$$0 \leq \int_a^b \frac{e^{-(N+1)x}}{1-e^{-x}}\, dx \leq e^{-Na} \int_a^b \frac{e^{-x}}{1-e^{-x}}\, dx = e^{-Na} \ln\left(\frac{1-e^{-b}}{1-e^{-a}}\right) \to 0 \text{ as } N \to \infty \text{ since } a > 0.$$

Thus we obtain:

$$\sum_{n=1}^{\infty} \frac{e^{-na} - e^{-nb}}{n} = \ln\left(\frac{1-e^{-b}}{1-e^{-a}}\right).$$

□