

## SOLUTION FOR OCTOBER 2024

A correct solution was submitted by:

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Find all continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$  such that

$$f(x+y) = f(x) + f(y) + f(x)f(y) \quad (1)$$

**SOLUTION** The solutions are  $f(x) = -1$  or  $f(x) = -1 + e^{cx}$  for some real number  $c$ .

**Proof** First we let  $g(x) = 1 + f(x)$ . Then it follows from (1) that

$$g(x+y) = 1 + f(x+y) = 1 + f(x) + f(y) + f(x)f(y) = (1 + f(x))(1 + f(y)) = g(x)g(y).$$

Thus

$$g(x+y) = g(x)g(y). \quad (2)$$

Next note that  $g(x) = g(\frac{x}{2} + \frac{x}{2}) = g(\frac{x}{2})g(\frac{x}{2}) = g^2(\frac{x}{2}) \geq 0$  and so  $g(x) \geq 0$ . Next if  $g(x_0) = 0$  for some  $x_0$  then  $g(x) = g(x_0)g(x - x_0) = 0$  for all  $x$  and therefore  $f(x) = -1$  for all  $x$ . Now we suppose  $g(x) > 0$  for all  $x$ . Then let  $h(x) = \ln(g(x))$  and we see from (2) that

$$h(x+y) = \ln(g(x+y)) = \ln(g(x)g(y)) = \ln(g(x)) + \ln(g(y)) = h(x) + h(y).$$

Thus

$$h(x+y) = h(x) + h(y). \quad (3)$$

and we also know  $h$  is continuous.

It is then well-known that  $h(x) = h(1)x$ . (First note that  $h(0) = h(0+0) = h(0) + h(0)$  and so  $h(0) = 0$ . Next note that  $h(2) = h(1) + h(1) = 2h(1)$ . With a bit more work one can show that  $h(r) = h(1)r$  for all rational  $r$  and then since  $h$  is assumed to be continuous then for general  $x$  write  $x = \lim_{n \rightarrow \infty} r_n$  where the  $r_n$  are rational and so by continuity of  $h$  we have  $h(x) = \lim_{n \rightarrow \infty} h(r_n) = \lim_{n \rightarrow \infty} h(1)r_n = h(1)x$ ).

So we now have

$$h(1)x = h(x) = \ln(g(x)) \text{ and so } g(x) = e^{h(1)x} \text{ thus } f(x) = -1 + e^{h(1)x}.$$

Letting  $c = h(1)$  we see then that

$$f(x) = -1 + e^{cx}.$$

Thus we see that the solutions are

$$f(x) = -1 \text{ or } f(x) = -1 + e^{cx} \text{ for some real } c.$$