**SOLUTION FOR DECEMBER 2024**

A correct solution was submitted by:

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**PROBLEM:** Suppose $f$ is continuous and $f(f(x)) = x$ for all $x$ and $f(0) = 0$, $f(1) = 1$. Prove $f(x) = x$ on $[0, 1]$.

**PROOF:** First it follows that $f$ is one-to-one for if $f(x_1) = f(x_2)$ then applying $f$ to both sides gives $x_1 = f(f(x_1)) = f(f(x_2)) = x_2$. Thus $f$ is one-to-one.

Now $f$ cannot have a local maximum or local minimum on $(0, 1)$ because if so then $f$ would not be one-to-one near that local extremum. Further since $f(0) = 0$ and $f(1) = 1$ and $f$ is continuous then it follows that $f$ must be increasing everywhere on $[0, 1]$.

We claim now that $f(x_1) = x_1$ for all $x_1$ in $(0, 1)$. So suppose not. Then there is an $x_1$ such that $f(x_1) \neq x_1$ and so without loss of generality let us suppose $f(x_1) > x_1$. Since $f$ is increasing then applying $f$ gives $x_1 = f(f(x_1)) > f(x_1)$ and thus $f(x_1) < x_1$ a contradiction. Thus $f(x_1) = x_1$ for all $x_1 \in (0, 1)$ and in fact on $[0, 1]$. $\square$