SOLUTION FOR JANUARY 2024

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Determine:

$$\lim_{n\to\infty}(n!e - [n!e])$$

Note: $[x]$ is the largest integer $\leq x$ so for example $[1.2] = 1, [\pi] = 3, [-1.2] = -2$).

Solution:

$$\lim_{n\to\infty}(n!e - [n!e]) = 0.$$

Proof: It follows from the Maclaurin series for $f(x) = e^x$ at $x = 1$ that:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \cdots .$$

Thus:

$$n!e = \left(n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{(n-1)!} + 1\right) + \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right).$$

Notice that the terms in the first set of parentheses are all integers. (For example $\frac{n!}{3!} = \frac{n!}{3!(n-3)!}(n-3)! = \binom{n}{3}(n-3)!$ where $\binom{n}{3}$ is the binomial coefficient which is an integer).

Next we claim that the second term in parentheses is a number that is greater than or equal to 0 and strictly less than 1 and therefore the above reads:

$$n!e = [n!e] + \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right).$$

Thus:

$$n!e - [n!e] = \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right).$$

To finish the proof we will now show that:

$$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right) = 0.$$

Notice that:

$$0 \leq \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right) \leq \left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right).$$

The series on the right is *geometric* and it is known for a *geometric series* with initial term a and common ratio r with $|r| < 1$ that:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

For the series we are considering we have $a = \frac{1}{n+1}$ and $r = \frac{1}{n+1} < 1$ and so its sum is:

$$\frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Therefore:

$$\lim_{n\to\infty} (n!e - [n!e]) = 0.$$

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