SOLUTION FOR FEBRUARY 2024

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February Problem:

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies:

$$f(f(x)) = -x \text{ for every } x \in \mathbb{R}. \tag{1}$$

Prove that f is one-to-one and onto. In addition, prove that any such function f cannot be continuous!

Solution: We first show f is one-to-one. So suppose $f(x) = f(y)$. Applying f again and (1) gives:

$$-x = f(f(x)) = f(f(y)) = -y$$

and thus $x = y$. Therefore f is one-to-one.

Next we show f is onto. So let $y_0 \in \mathbb{R}$. Now let $x_0 = f(-y_0)$. Then using (1) we see $f(x_0) = f(f(-y_0)) = y_0$ and thus f is onto.

Finally we suppose f is continuous and try to obtain a contradiction.

Using (1) we see $f(f(0)) = 0$ and so let us denote $f(0) = a$. Then we see $f(a) = f(f(0)) = 0$. Without loss of generality let us suppose $a \geq 0$. We now claim that $f(0) = a = 0$ for suppose $a > 0$ then we see that since f is continuous then $g(x) = f(x) - x$ is continuous. Also $g(0) = f(0) - 0 = a - 0 > 0$ and $g(a) = f(a) - a = 0 - a < 0$. Then by the Intermediate Value Theorem there is a b with $0 < b < a$ such that $g(b) = 0$, that is, $f(b) = b$. Applying f and using (1) gives $-b = f(f(b)) = f(b)$. Therefore $-b = f(b) = b$ and thus $b = 0$ and therefore $f(0) = 0$ - a contradiction. Therefore we see that $f(0) = 0$.

Next since f is continuous and one-to-one we must have either $f(x) > 0$ for $x > 0$ or $f(x) < 0$ for $x > 0$. (To see this suppose there are positive $c_1 < c_2$ with $f(c_1) > 0$ and $f(c_2) < 0$. Then by the Intermediate Value Theorem there is a c_3 with $c_1 < c_3 < c_2$ such that $f(c_3) = 0$ but since f is one-to-one and $f(0) = 0$ then this forces $c_3 = 0$ which contradicts that $c_3 > 0$). So without loss of generality let us assume $f(x) > 0$ for $x > 0$. Now let us denote $f(1) = c > 0$. Then by (1) we see $-1 = f(f(1)) = f(c)$. But this contradicts that $f(x) > 0$ for $x > 0$. Therefore we see that any such function must be discontinuous.

Note: In fact it can be shown that f must have an infinite number of discontinuities! $\square$