SOLUTION FOR APRIL 2024

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Let $i = \sqrt{-1}$. Determine whether the following infinite products converge or diverge:

$$\prod_{n=1}^{\infty} \left(1 + \frac{i}{n}\right) \text{ and } \prod_{n=1}^{\infty} |1 + \frac{i}{n}|.$$

Solution:

$$\prod_{n=1}^{\infty} \left(1 + \frac{i}{n}\right) \text{ diverges and } \prod_{n=1}^{\infty} |1 + \frac{i}{n}| \text{ converges.}$$

Proof: We first examine:

$$\prod_{n=1}^{\infty} |1 + \frac{i}{n}| = \prod_{n=1}^{\infty} \sqrt{1 + \frac{1}{n^2}}.$$

Let:

$$p_N = \prod_{n=1}^{N} \sqrt{1 + \frac{1}{n^2}}.$$

Then:

$$\ln(p_N) = \frac{1}{2} \sum_{n=1}^{N} \ln\left(1 + \frac{1}{n^2}\right). \tag{1}$$

We now use the following inequality which holds for all $x \geq 0$:

$$0 \leq \ln(1 + x) \leq x.$$

(To prove this let $x \geq 0$ and $g(x) = x - \ln(1+x)$. Then $g(0) = 0$ and $g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \geq 0$ from which it follows that $g(x) \geq 0$.) Using this inequality above gives:

$$0 \leq \ln(p_N) \leq \frac{1}{2} \sum_{n=1}^{N} \frac{1}{n^2} = \text{ this is a } p \text{ series with } p = 2 > 1 \text{ and so } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

Thus we see from (1) that $\ln(p_N)$ is increasing and bounded above and thus $\lim_{N\to\infty} \ln(p_N)$ exists which implies $\lim_{N\to\infty} p_N$ exists. Thus:

$$\prod_{n=1}^{\infty} |1 + \frac{i}{n}| \text{ converges.}$$

For the next part it helps to rewrite $1 + \frac{i}{n}$ in polar form. In fact:

$$1 + \frac{i}{n} = \sqrt{1 + \frac{1}{n^2}}\, e^{i \tan^{-1}\left(\frac{1}{n}\right)}.$$

Then by rules of exponentials:

$$\prod_{n=1}^{N}\left(1+\frac{i}{n}\right)=\prod_{n=1}^{N}\sqrt{1+\frac{1}{n^2}}e^{i\sum_{n=1}^{N}\tan^{-1}\left(\frac{1}{n}\right)}.$$

Now the product on the right converges by the first part of this problem so we just need to determine:

$$\text{whether or not } \sum_{n=1}^{N}\tan^{-1}\left(\frac{1}{n}\right) \text{ converges.}$$

We now use the following inequality which holds for all $0 \leq x \leq 1$:

$$\tan^{-1}(x) \geq \frac{1}{2}x.$$

To see this let $h(x) = \tan^{-1}(x) - \frac{1}{2}x$. Then notice that $h(0) = 0$ and $h'(x) = \frac{1}{1+x^2} - \frac{1}{2} \geq 0$ for $0 \leq x \leq 1$ and thus $h(x) \geq 0$ for $0 \leq x \leq 1$.
It follows from this that:

$$\sum_{n=1}^{N}\tan^{-1}\left(\frac{1}{n}\right) \geq \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} \text{ and as is well-known } \sum_{n=1}^{N}\frac{1}{n} \to \infty \text{ as } N \to \infty.$$

Therefore we see:

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{n}\right) \text{ diverges}$$

and therefore:

$$\prod_{n=1}^{\infty}\left(1+\frac{i}{n}\right) \text{ diverges.}$$

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