

SOLUTION FOR SEPTEMBER 2024

The only correct solution was submitted by

Winner: Saikiran Motati

Let a, b be positive. Prove that:

$$G(a, b) = G\left(\frac{a+b}{2}, \sqrt{ab}\right)$$

where:

$$G(a, b) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} dx.$$

Proof: We change variables and set:

$$x = \frac{1}{2} \left(u - \frac{ab}{u} \right) \text{ where } 0 < u < \infty.$$

Then $-\infty < x < \infty$ and:

$$dx = \frac{1}{2} \left(1 + \frac{ab}{u^2} \right) du = \frac{u^2 + ab}{2u^2} du.$$

Also:

$$\sqrt{x^2 + ab} = \frac{1}{2} \left(u + \frac{ab}{u} \right) = \frac{u^2 + ab}{2u}$$

and:

$$\sqrt{x^2 + \left(\frac{a+b}{2} \right)^2} = \frac{\sqrt{u^2 + a^2} \sqrt{u^2 + b^2}}{2u}.$$

Then we see:

$$\begin{aligned} G\left(\frac{a+b}{2}, \sqrt{ab}\right) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{(x^2 + (\frac{a+b}{2})^2)(x^2 + ab)}} dx \\ &= 2 \int_0^{\infty} \frac{1}{\sqrt{(u^2 + a^2)(u^2 + b^2)}} du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{(u^2 + a^2)(u^2 + b^2)}} du = G(a, b). \end{aligned}$$

□

Note: Given $a_0 > b_0 > 0$ we define:

$$a_{n+1} = \frac{a_n + b_n}{2}; b_{n+1} = \sqrt{a_n b_n}.$$

This is called the *arithmetic-geometric mean sequence* of a_0 and b_0 . It turns out it can be shown that a_n is decreasing and b_n is increasing for $n \geq 1$ and in fact there is a $c > 0$ such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = c$!!! Thus from the problem just solved we see that:

$$G(a_{n+1}, b_{n+1}) = G\left(\frac{a_n + b_n}{2}, \sqrt{a_n b_n}\right) = G(a_n, b_n)$$

and therefore:

$$G(a_n, b_n) = G(a_0, b_0) \text{ for all } n \geq 1.$$

Further since $G(a, b)$ is continuous in both a and b then taking the limit as $n \rightarrow \infty$ we obtain:

$$G(c, c) = G(a_0, b_0)$$

but $G(c, c)$ is easy to compute and it is just $G(c, c) = \frac{\pi}{c}$. Thus we see that:

$$c = \frac{\pi}{G(a_0, b_0)}.$$

The number c is often denoted $c = AGM(a_0, b_0)$.