SOLUTION FOR OCTOBER 2025

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Problem:

Determine

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\cdots.$$

HINT: First show

$$\frac{1}{2}\int_0^1 t^{n-1}(1-t)^2\,dt=\frac{1}{n(n+1)(n+2)}.$$

Solution:

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\cdots=-\frac{1}{2}+\ln(2).$$

Proof: The hint can be shown by integrating by parts twice. Thus:

$$\frac{1}{2}\int_0^1 t^{n-1}(1-t)^2\,dt=\frac{1}{2}\frac{t^n}{n}(1-t)^2|_0^1+\int_0^1\frac{t^n}{n}(1-t)\,dt$$

$$=\frac{t^{n+1}}{n(n+1)}(1-t)|_0^1+\int_0^1\frac{t^{n+1}}{n(n+1)}\,dt=\frac{1}{n(n+1)(n+2)}.$$

Next we use this identity and sum over the odd integers with $n=2k+1$. This gives:

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\cdots=\sum_{k=0}^{\infty}\frac{1}{2}\int_0^1 t^{2k}(1-t)^2\,dt.$$

Next we note that $\sum\limits_{k=0}^{\infty} t^{2k}=\frac{1}{1-t^2}$ for $|t|<1$ and so we obtain

$$\frac{1}{2}\int_0^1\sum_{k=0}^{\infty}t^{2k}(1-t)^2\,dt=\frac{1}{2}\int_0^1\frac{(1-t)^2}{1-t^2}\,dt=\frac{1}{2}\int_0^1\frac{1-t}{1+t}\,dt$$

$$=\int_0^1\left(-\frac{1}{2}+\frac{1}{1+t}\right)\,dt=\left(-\frac{1}{2}t+\ln(1+t)\right)|_0^1=-\frac{1}{2}+\ln(2).$$

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