SOLUTION FOR NOVEMBER 2025

Problem: The Cissoid of Diocles - The "Duplication of the Cube" Problem

Let $a > 0$. The Cissoid in Cartesian coordinates is

$$y^2 = \frac{x^3}{a - x}.$$

Let $A = (a, 0)$ and $B = (0, 2a)$. Draw the line from A to B and denote P as the interesction point of this line and the cissoid. Draw the line from the origin through P to where it intersects with the line $x = a$. Denote this point Q. Prove $(AQ)^3 = 2(OA)^3$ and prove then that the cube of a has been doubled. (AQ is the distance from A to Q and OA is the distance from O to A).

Proof: The line from A to B is given by $y = 2(a - x)$. The intersection of this line and the cissoid occurs when $2(a - x) = \frac{x^{3/2}}{(a-x)^{1/2}}$. Solving for x gives $x = \frac{\sqrt[3]{4}\, a}{\sqrt[3]{4}+1}$ and $y = \frac{2a}{\sqrt[3]{4}+1}$. Thus

$$P = \left(\frac{\sqrt[3]{4}\, a}{\sqrt[3]{4} + 1}, \frac{2a}{\sqrt[3]{4} + 1} \right).$$

Now the line through the origin and P has slope $m = \frac{\frac{2a}{\sqrt[3]{4}+1}}{\frac{\sqrt[3]{4}\, a}{\sqrt[3]{4}+1}} = \frac{2}{\sqrt[3]{4}} = \sqrt[3]{2}$ and so the line through the origin and P is given by $y = \sqrt[3]{2}\, x$.

Finally, when $x = a$ this gives $y = \sqrt[3]{2}\, a$. So the point Q is given by $Q = (a, \sqrt[3]{2}\, a)$. Then we see $AQ = \sqrt[3]{2}\, a$ and $OA = a$. Thus $(AQ)^3 = 2a^3 = 2(OA)^3$. $\square$