SOLUTION FOR JANUARY 2025

A correct solution was turned in by:

Anirudh Mazumder

PROBLEM Describe the set S which is the set of all ordered pairs (a, b) of real numbers such that both (possibly complex) roots of $z^2 + az + b = 0$ satisfy $|z| < 1$.

SOLUTION

$$S = \{(a, b)|\ |a| - 1 < b < 1\}.$$

Proof Suppose z solves $z^2 + az + b = 0$ and suppose both roots of this satisfy $|z| < 1$. Then by the quadratic formula we have

$$z = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

CASE 1 $a^2 - 4b < 0$

In this case we see that

$$z = \frac{-a \pm i\sqrt{4b - a^2}}{2}$$

and thus

$$|z|^2 = \frac{a^2 + 4b - a^2}{4} = b$$

so in this case we see that $|z| < 1$ if and only if $\frac{a^2}{4} < b < 1$.
Now notice that $\frac{a^2}{4} < b < 1$ thus $a^2 < 4b$ and therefore $0 \leq (a \pm 2)^2 = a^2 \pm 4a + 4 < 4b \pm 4a + 4$. Thus $(\pm a) - 1 < b$ and therefore $|a| - 1 < b < 1$.

CASE 2 $a^2 - 4b \geq 0$

Subcase 2i. $a \geq 0$

Here we have $z_1 = \frac{-a - \sqrt{a^2 - 4b}}{2} \leq \frac{-a + \sqrt{a^2 - 4b}}{2} = z_2 \leq 0$. Thus if we want $|z| < 1$ then we must have $|\frac{-a - \sqrt{a^2 - 4b}}{2}| < 1$. That is,

$$\frac{a + \sqrt{a^2 - 4b}}{2} < 1.$$

Thus

$$0 \leq \sqrt{a^2 - 4b} < 2 - a \tag{1}$$

and therefore $a < 2$. And since we are in Case 2 this implies $4b \leq a^2 < 4$ and thus $b < 1$.
Now squaring (1) gives $a^2 - 4b < 4 - 4a + a^2$ hence $-4b < 4 - 4a$ thus

$$a - 1 < b.$$

Subcase 2ii. $a < 0$

Replacing a with $-a$ in subcase 2i gives

$$-a - 1 < b$$

so combining Subcase 2i, 2ii and Case 1 gives

$$|a| - 1 < b.$$

Thus if $z^2 + az + b = 0$ has both solutions with $|z| < 1$ then $(a, b) \in S$.

Conversely, suppose $(a, b) \in S$. Then we want to show that the two solutions of $z^2 + az + b = 0$ satisfy $|z| < 1$. So let us suppose $|a| - 1 < b < 1$.

CASE 1 $a^2 - 4b < 0$

In this case as we saw earlier then

$$|z|^2 = \frac{a^2 + 4b - a^2}{4} = b$$

and since we are assuming $b < 1$ it follows that $|z| < 1$.

CASE 2 $a^2 - 4b \geq 0$

So now again suppose $|a| - 1 < b < 1$. Then it follows that $|a| < 2$ and also we see that $4|a| - 4 < 4b$ so $-4|a| + 4 > -4b$ and thus $(2 - |a|)^2 = a^2 - 4|a| + 4 > a^2 - 4b$. Taking roots and recalling $|a| < 2$ we obtain $2 > |a| + \sqrt{a^2 - 4b}$. Thus we see that $\frac{|a|+\sqrt{a^2-4b}}{2} < 1$. Hence $|z| = |\frac{-a\pm\sqrt{a^2-4b}}{2}| \leq \frac{|a|+\sqrt{a^2-4b}}{2} < 1$. $\square$