SOLUTION FOR APRIL 2025

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Problem: Let $f : \mathbb{R} \to \mathbb{R}$ satisfy:

$$f(x+y) + f(x-y) = 2f(x)f(y) \text{ and } f(1) = -1. \tag{1}$$

Show that f is periodic. That is, show there is a $p \neq 0$ such that $f(x+p) = f(x)$ for all x.

Solution:

Let $x = y = 0$ which gives $2f(0) = 2f^2(0)$ so $f(0) = 0$ or $f(0) = 1$. Now let $y = 0$ and we get $2f(x) = 2f(x)f(0)$. If $f(0) = 0$ then $f(x) \equiv 0$ but this contradicts that $f(1) = -1$. Therefore we see $f(0) = 1$.

Next let $x = y$ to obtain $f(2x) + f(0) = 2f^2(x)$. Thus $f(2x) = 2f^2(x) - 1$. Now let $x = \frac{1}{2}$ to obtain $-1 = f(1) = 2f^2(\frac{1}{2}) - 1$. Thus $f(\frac{1}{2}) = 0$.

Then using (1) and $y = \frac{1}{2}$ we see that $f(x + \frac{1}{2}) + f(x - \frac{1}{2}) = 0$ so $f(x - \frac{1}{2}) = -f(x + \frac{1}{2})$. Now let $y = x - \frac{1}{2}$ then $y + 1 = y + 2(\frac{1}{2}) = x + \frac{1}{2}$ and thus $f(y+1) = -f(y)$. Then $f(y+2) = -f(y+1) = f(y)$ and thus $f(y+2) = f(y)$ so f is periodic.