## SOLUTION FOR MAY 2025

Determine all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = y + f^2(x). \qquad (1)$$

**Solution:** $f(x) = 0$ for all $x$ or $f(x) = x$ for all $x$.

**PROOF:** Let $x = 0$ then we obtain

$$f(f(y)) = y + f^2(0). \qquad (2)$$

This implies $f$ is one-to-one because if $f(a) = f(b)$ then $a + f^2(0) = f(f(a)) = f(f(b)) = b + f^2(0)$ and so $a = b$.
Also, $f$ is onto because given $y$ then $f(f(y - f^2(0))) = y - f^2(0) + f^2(0) = y$.

Thus $f$ is one-to-one and onto. Thus there exists a $y_0$ such that $f(y_0) = 0$.

Substituting $x = y = y_0$ into (1) gives $f(y_0^2) = f(y_0^2+f(y_0)) = y_0+f^2(y_0) = y_0+0 = y_0$. Thus $f(y_0^2) = y_0$ so using (2) we see $y_0^2 + f^2(0) = f(f(y_0^2)) = f(y_0) = 0$. Thus $y_0 = 0$ so we see $f(0) = 0$ and then (2) becomes

$$f(f(y)) = y \qquad (3).$$

Now let $y = 0$ in (1) to obtain:

$$f(x^2) = f^2(x).$$

Replace $x$ by $-x$ to get

$$f^2(x) = f(x^2) = f((-x)^2) = f^2(-x).$$

Thus either $f(-x) = f(x)$ or $f(-x) = -f(x)$. If $x \neq 0$ then $f(-x) = f(x)$ is false because $f$ is one-to-one so it must be that

$$f(-x) = -f(x).$$

Also let $x = 1$ above and we get $f(1) = f^2(1)$ so $f(1) = 1$ or $f(1) = 0$. But $f(1) \neq 0$ because $f$ is one-to-one and $f(0) = 0$. Thus $f(1) = 1$.

Now replace $y$ with $f(y)$ in (1) to obtain

$$f(x^2 + y) = f(x^2 + f(f(y))) = f(y) + f^2(x) = f(x^2) + f(y).$$

Let $z = x^2$ and we see if $z \geq 0$ then

$$f(z + y) = f(z) + f(y).$$

Now suppose $z < 0$ then using the above and that $f$ is odd then

$$-f(z - y) = f((-z) + y) = f(-z) + f(y) = -f(z) + f(y)$$

thus

$$f(z+(-y)) = f(z-y) = f(z) - f(y) = f(z) + f(-y)$$

and letting $w = -y$ we get

$$f(z+w) = f(z) + f(w)$$

and so $f$ is linear. This implies $f(r) = r$ for rational $r$.

Next suppose $z > y$ then $z - y = c^2$ so

$$0 \le f(c)^2 = f(c^2) = f(z-y) = f(z) - f(y).$$

Thus $f$ is an increasing function.

Finally let $x \in \mathbb{R}$ and choose $r_n, q_n$ rational with $r_n < r_{n+1} < x < q_{n+1} < q_n$ such that $\lim_{n\to\infty} r_n = x = \lim_{n\to\infty} q_n$. Then $r_n = f(r_n) \le f(x)$ since $r_n < x$. And since $r_n \to x$ then this implies $x \le f(x)$. Similarly, $f(x) \le f(q_n) = q_n$ and since $q_n \to x$ then we see $f(x) \le x$. Thus $f(x) = x$ for all $x$. □