SOLUTION FOR SEPTEMBER 2025

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Problem:

Derive Newton's series:

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\cdots=\frac{\pi}{2\sqrt{2}}.$$

Hint: Evaluate $\int_0^1 \frac{1+x^2}{1+x^4}\,dx$ in two different ways.

Proof: One way to evaluate this integral is using power series.

Recall that for $|x|<1$ we have

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots.$$

Replacing x with $-x^4$ gives that for $|x|<1$

$$\frac{1}{1+x^4}=1-x^4+x^8-x^{12}+x^{16}+\cdots.$$

Multiplying by x^2 gives

$$\frac{x^2}{1+x^4}=x^2-x^6+x^{10}-x^{14}+x^{18}+\cdots.$$

Adding these two gives

$$\frac{1+x^2}{1+x^4}=1+x^2-x^4-x^6+x^8+x^{10}-x^{12}-x^{14}+\cdots.$$

Integrating on $[0,1]$ gives

$$\int_0^1 \frac{1+x^2}{1+x^4}\,dx=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\cdots. \tag{1}$$

Another way to rewrite the integral is by dividing each term by x^2 to obtain

$$\int_0^1 \frac{1+x^2}{1+x^4}\,dx=\int_0^1 \frac{\frac{1}{x^2}+1}{x^2+\frac{1}{x^2}}\,dx.$$

Now make the substitution $u=x-\frac{1}{x}$. This gives $u^2+2=x^2+\frac{1}{x^2}$ and $du=(1+\frac{1}{x^2})\,dx$. So we obtain

$$\int_{-\infty}^0 \frac{1}{u^2+2}\,du=\frac{1}{\sqrt{2}}\tan^{-1}(\frac{u}{\sqrt{2}})|_{-\infty}^0=0-\frac{1}{\sqrt{2}}(-\frac{\pi}{2})=\frac{\pi}{2\sqrt{2}}. \tag{2}$$

Equating (1) and (2) yields the result. □