SOLUTION FOR JANUARY 2026

Problem: Let n be a positive integer and let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Next let $S(N)$ be the smallest value of n such that $H_n > N$. Prove that

$$\lim_{N\to\infty} \frac{S(N+1)}{S(N)} = e.$$

HINT: It is known that

$$\lim_{n\to\infty} (H_n - \ln(n)) = \gamma \tag{1}$$

where $\gamma \in \mathbb{R}$ is called the *Euler-Mascheroni constant* and $\gamma \approx .57$.

Proof: First notice that it follows from (1) that $\lim_{N\to\infty} S(N) = \infty$.

Let us denote $S(N) = n_0$. Then it follows that

$$H_{n_0} > N \geq H_{n_0-1}. \tag{2}$$

Similarly denote $S(N+1) = n_1$ and so

$$H_{n_1} > N + 1 \geq H_{n_1-1}. \tag{3}$$

Multiplying (2) by (-1) and rewriting gives

$$-H_{n_0} < -N \leq -H_{n_0-1}. \tag{4}$$

Adding (3) and (4) gives

$$H_{n_1-1} - H_{n_0} < 1 < H_{n_1} - H_{n_0-1}. \tag{5}$$

Next we rewrite the left-hand inequality in (5) as

$$(H_{n_1-1} - \ln(n_1 - 1)) - (H_{n_0} - \ln(n_0)) + (\ln(n_1 - 1) - \ln(n_0)) < 1.$$

Thus

$$(H_{n_1-1} - \ln(n_1 - 1)) - (H_{n_0} - \ln(n_0)) + \ln\left(\frac{S(N+1) - 1}{S(N)}\right) < 1.$$

Let us denote $K(n) = H_n - \ln(n)$ and then we may rewrite the above as

$$\ln\left(\frac{S(N+1) - 1}{S(N)}\right) < 1 + K(n_0) - K(n_1 - 1) = 1 + K(S(N)) - K(S(N+1) - 1)$$

and so

$$\frac{S(N+1)}{S(N)} < \frac{1}{S(N)} + e^{1+K(S(N))-K(S(N+1)-1)}.$$

Similarly rewriting the right-hand inequality in (5) yields

$$\frac{S(N+1)}{S(N)} > \left(1 - \frac{1}{S(N)}\right) e^{1-K(S(N+1))+K(S(N)-1)}.$$

Thus

$$\left(1 - \frac{1}{S(N)}\right) e^{1-K(S(N+1))+K(S(N)-1)} < \frac{S(N+1)}{S(N)} < \frac{1}{S(N)} + e^{1+K(S(N))-K(S(N+1)-1)}. \quad (6)$$

Recall from earlier that $S(N) \to \infty$ as $N \to \infty$. Also it follows from the hint that

$$K(S(N)) \to \gamma \text{ as } N \to \infty.$$

This then implies

$$K(S(N+1)) \to \gamma,\ K(S(N)-1) \to \gamma, \text{ and } K(S(N+1)-1) \to \gamma \text{ as } N \to \infty.$$

It follows then that the left-hand side and right-hand side of (6) converge to e as $N \to \infty$. Thus $\lim_{N\to\infty} \frac{S(N+1)}{S(N)} = e$. □