SOLUTION FOR FEBRUARY 2026

A correct solution was submitted by

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Problem:

Determine what happens to the sequence of numbers

$$\sqrt{7}, \sqrt{7-\sqrt{7}}, \sqrt{7-\sqrt{7+\sqrt{7}}}, \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}, \cdots.$$

Solution: Denoting this sequence as x_n then $\lim_{n\to\infty} x_n = 2$.

Proof: Notice that

$$x_{n+2} = \sqrt{7-\sqrt{7+x_n}} \tag{1}$$

with $x_1 = \sqrt{7}$ and $x_2 = \sqrt{7-\sqrt{7}}$. Thus we can write this sequence as $x_{n+2} = f(x_n)$ where

$$f(x) = \sqrt{7-\sqrt{7+x}}.$$

Note that

$$f : [-7, 42] \to \mathbb{R} \text{ and } 0 \leq f(x) \leq \sqrt{7}. \text{ In addition, } f(2) = 2 \text{ and } f'(x) < 0 \text{ on } (-7, 42].$$

Thus since $x_1 = \sqrt{7}$ and $x_2 = \sqrt{7-\sqrt{7}}$ (which are both in $[-7, 42]$) then x_3 and x_4 are in $[0, \sqrt{7}] \subset [-7, 42]$ and therefore x_n is defined for all $n \geq 1$.

Next observe by applying the mean value theorem twice we see that for some c_1 and c_2 in $[0, \sqrt{7}]$ that

$$x_{n+4} - x_n = f(x_{n+2}) - f(x_{n-2}) = f'(c_1)(x_{n+2} - x_{n-2}) = f'(c_1)f'(c_2)(x_n - x_{n-4}) \text{ and } f'(c_1)f'(c_2) > 0.$$

Therefore if $x_n < x_{n-4}$ then $x_{n+4} < x_n$.

And it is not that hard to show $x_5 < x_1$. Therefore, it follows that the subsequence x_{4k+1} is decreasing, positive, and therefore has a limit. Thus

$$\lim_{k\to\infty} x_{4k+1} = L_1 \text{ and } 0 \leq L_1 \leq \sqrt{7}.$$

In a similar way it follows that x_{4k+2}, x_{4k+3}, and x_{4k+4} are monotone and bounded and so they have limits L_2, L_3, L_4 $\lim_{k\to\infty} x_{4k+2} = L_2$ $\lim_{k\to\infty} x_{4k+3} = L_3$ $\lim_{k\to\infty} x_{4k+4} = L_4$ with each L_i between 0 and $\sqrt{7}$.

Next we show $L_2 = L_4$. Notice that

$$x_{n+4} - x_{n+2} = \sqrt{7 - \sqrt{7 + x_{n+2}}} - \sqrt{7 - \sqrt{7 + x_n}} = \frac{\sqrt{7 + x_n} - \sqrt{7 + x_{n+2}}}{\sqrt{7 - \sqrt{7 + x_{n+2}}} + \sqrt{7 - \sqrt{7 + x_n}}}$$

$$= \frac{-(x_{n+2} - x_n)}{(\sqrt{7 - \sqrt{7 + x_{n+2}}} + \sqrt{7 - \sqrt{7 + x_n}})(\sqrt{7 + x_n} + \sqrt{7 + x_{n+2}})}.$$

Thus letting $n = 4k$ we obtain

$$x_{4k+4} - x_{4k+2} = \frac{-(x_{4k+2} - x_{4k})}{(\sqrt{7 - \sqrt{7 + x_{4k+2}}} + \sqrt{7 - \sqrt{7 + x_{4k}}})(\sqrt{7 + x_{4k}} + \sqrt{7 + x_{4k+2}})}.$$

Now taking limits gives

$$L_4 - L_2 = \frac{-(L_2 - L_4)}{(\sqrt{7 - \sqrt{7 + L_2}} + \sqrt{7 - \sqrt{7 + L_4}})(\sqrt{7 + L_4} + \sqrt{7 + L_2})}$$

from which it follows that $L_2 = L_4$. A similar argument shows $L_1 = L_3$.

Next, letting $n = 4k$ and taking limits in (1) gives

$$L_4 = \sqrt{7 - \sqrt{7 + L_4}}. \tag{2}$$

Thus $\sqrt{7 + L_4} = 7 - L_4^2$ and therefore

$$(L_4 - 2)(L_4 + 3)(L_4^2 - L_4 - 7) = L_4^4 - 14L_4^2 - L_4 + 42 = 0$$

and so

$$L_4 = 2, -3, \frac{1 \pm \sqrt{29}}{2}.$$

Since $0 \leq L_4 \leq \sqrt{7}$ we must have $L_2 = L_4 = 2$.

Similarly L_3 satisfies (2) and similarly it can be shown that $L_1 = L_3 = 2$ and hence $L_1 = L_2 = L_3 = L_4 = 2$. $\square$