Math 1720. Solving equations involving ln.

Since the book doesn't cover this topic very well, and I didn't get to finish the second example in class on Friday 1/20, I give 2 examples here (one of which was the second example from Friday). I'll also go over finishing that example in class on Monday.

Summary: in dealing with equations involving ln, we use the various algebraic properties (on page 389 of the text), and the fact that for $a, b > 0$, $\ln(a) = \ln(b)$ if and only if $a = b$. We also have to pay careful attention to the fact that $\ln(x)$ is only defined for $x > 0$.

Example 1:
Find all solutions x to the equation

$$\ln(4x) - \ln(3) = \ln(x - 5) + \ln(2).$$

First note that $\ln(y)$ is only defined for $y > 0$. So $\ln(4x)$ is only defined when $4x > 0$, i.e. when $x > 0$. And $\ln(x - 5)$ is only defined when $x - 5 > 0$, i.e. $x > 5$. So we need both $x > 0$ and $x > 5$, but since $5 > 0$, this is equivalent to just $x > 5$.

Now we look for solutions: First using the algebraic property $\ln(x/y) = \ln(x) - \ln(y)$,

$$\ln(4x) - \ln(3) = \ln(4x/3).$$

And the property $\ln(xy) = \ln(x) + \ln(y)$ gives

$$\ln(x - 5) + \ln(2) = \ln(2(x - 5)).$$

So the original equation is equivalent to

$$\ln(4x/3) = \ln(2(x - 5)), \quad x > 5.$$

(I've appended the requirement $x > 5$ there since this was the domain restriction for the original equation. When making algebraic manipulations, sometimes the original domain restriction can otherwise get lost. In other words it's a reminder that at the end, we have to check that any solutions we find satisfy $x > 5$.)

Now since $\ln(x) = \ln(y)$ iff $x = y$ (for $x, y > 0$), we get:

$$4x/3 = 2(x - 5), \quad x > 5.$$

So

$$4x = 6(x - 5), \quad x > 5.$$

$$-2x = -30, \quad x > 5.$$

$$x = -30/(-2) = 15, \quad x > 5.$$

So $x = 15$ is the only possible solution, and since $15 > 5$, it is indeed a (and the only) solution.

Example 2 (started Friday): find all solutions to the equation

$$\ln(x + 2) = 2\ln(1 - x) - \ln(2).$$

Solution. First note that $\ln(x+2)$ is only valid for $x+2>0$, i.e. $x>-2$.

And $\ln(1-x)$ is only valid for $1-x>0$, i.e. $x<1$. So any solutions x must be in the interval

$$-2<x<1.$$

Now we look for solutions.

Applying the rule $\ln(x^r)=r\ln(x)$ to the right side of the equation:

$$2\ln(1-x)-\ln(2)=\ln((1-x)^2)-\ln(2),$$

and applying the rule $\ln(x/y)=\ln(x)-\ln(y)$:

$$=\ln(\frac{1}{2}(1-x)^2)$$

So the original equation is equivalent to

$$\ln(x+2)=\ln(\frac{1}{2}(1-x)^2),\quad -2<x<1.$$

This equation has the form

$$\ln(a)=\ln(b),$$

so it is equivalent to $a=b$, i.e.

$$x+2=\frac{1}{2}(1-x)^2,\quad -2<x<1.$$

Rearranging and solving:

$$2x+4=1-2x+x^2,\quad -2<x<1.$$

$$x^2-4x-3=0,\quad -2<x<1.$$

Using the quadratic formula:

$$\begin{aligned} x &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-3)}}{2(1)} \\ &= \frac{4\pm\sqrt{16+12}}{2} \\ &= \frac{4\pm\sqrt{28}}{2} = 2\pm\frac{2\sqrt{7}}{2} \\ &= 2\pm\sqrt{7}. \end{aligned}$$

So we have two possible solutions, $x=2\pm\sqrt{7}$, but we also have the restriction $-2<x<1$. The possible solution $x=2+\sqrt{7}$ is >1, so it is not valid. The possible solution $x=2-\sqrt{7}$ is valid:

$$-2<2-\sqrt{7}<1,$$

because $\sqrt{7}$ is between 2 and 3: This is because $4<7<9$, so $\sqrt{4}<\sqrt{7}<\sqrt{9}$, so $2<\sqrt{7}<3$. So we get

$$2-3<2-\sqrt{7}<2-2,$$

i.e.

$$-1<2-\sqrt{7}<0,$$

so certainly

$$-2<2-\sqrt{7}<1.$$

So $x=2-\sqrt{7}$ is the unique solution.