Measures in Mice

Farmer Schlutzenberg

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Dissertation Advisor: John Steel

Abstract

This thesis analyses extenders appearing in fine structural mice. Kunen showed that in the inner model for one measurable cardinal, there is a unique normal measure. This result is generalized, in various ways, to mice below a superstrong cardinal.

The analysis is then used to show that certain tame mice satisfy $\mathsf{V} = \mathsf{HOD}$. In particular, the approach provides a new proof of this result for the inner model M_n for n Woodin cardinals. It is also shown that in M_n, all homogeneously Suslin sets of reals are $\mathbf{\Delta}^1_{n+1}$.

Contents

1 Introduction

In [5], Kunen showed that if $V = L[U]$ where U is a normal measure, then U is the unique normal measure, and all measures are reducible to finite products of this normal measure. Mitchell constructed inner models with sequences of measurables in [8] and [9], and proved related results characterizing measures in those models. Dodd similarly characterized the extenders appearing in his inner models for strong cardinals, in [1]. The first few sections of this thesis extend these results to inner models below a superstrong cardinal. However, the models we deal with are fine structural premice as in [22].

Given a mouse N satisfying "E is a total, wellfounded extender", we are interested in how E relates to N's extender sequence $\mathbb{E}^N$. In particular, we would like to know whether E is on $\mathbb{E}^N$, or on the sequence of an iterate, or more generally, whether E is the extender of an iteration map on N. Although we have only a partial understanding, we do, happily, have some affirmative results.

Some of the following theorems are stated without any smallness assumption on the mice involved. However, the premice we work with do not have extenders of superstrong type indexed on their sequence (see [22]). Removing this restriction, a counterexample to 3.7 is soon reached, though it seems the statement of the theorem might be adapted to deal with this. In the following, ν_E denotes the natural length (or support) of an extender E. (See the end of this introduction for notation and definitions.)

Corollary (2.10 (Steel, Schlutzenberg))**.** *Let N be an $(\omega, \omega_1, \omega_1{+}1)$-iterable mouse satisfying* ZFC*, and suppose*

$$N \models E \textit{ is a short, total extender, } \nu = \nu_E \textit{ is a cardinal, and } \mathcal{H}_\nu \subseteq \mathrm{Ult}(V, E).$$

Then the trivial completion of E is on $\mathbb{E}^N$.

Here and below, one can make do with much less than ZFC (see 2.10). A stronger theorem is actually proven (see 2.9). Notice that if E is a normal measure in N, then $\nu_E = (\mathrm{crit}(E)^+)^N$ is an N-cardinal, and $\mathcal{H}^N_{\nu_E} \subseteq \mathrm{Ult}(N, E)$. As a corollary to the proof, we'll also obtain that if N is a mouse modelling ZFC, and κ is uncountable in N, then $L(\mathcal{P}(\kappa) \cap N) \models \mathsf{AC}$. This confirms a conjecture of Hugh Woodin. The hypothesis "κ uncountable in N" is necessary, since Woodin proved that $\mathsf{AD}^{L(\mathbb{R})}$ holds assuming there are ω Woodin cardinals with a measurable above (see [13]).

Theorem (3.7)**.** *Let N be an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse satisfying* ZFC*, and suppose in N, E is a wellfounded extender, which is its own trivial completion, and $(N||\mathrm{lh}(E), E)$ is a premouse. Then E is on $\mathbb{E}^N$.*

Again a stronger theorem is proven, in which E fits on the sequence of an iterate of N instead.

An extender F (possibly partial) is *finitely generated* if there is $s \in [\nu_F]^{<\omega}$ so that for each $\alpha < \nu_F$, there's f satisfying $\alpha = [s, f]_F$.

Theorem (4.8)**.** *Let N be an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse satisfying* ZFC*, and suppose in N, E is a countably complete ultrafilter. Then there is a finite normal (fine-structural) iteration tree $\mathcal{T}$ on N, with last model* $\mathrm{Ult}(N, E)$*, and i^N_E is the main branch embedding of $\mathcal{T}$. Moreover, $\mathcal{T}$'s extenders are all finitely generated.*

In §4 we also investigate partial measures: if N is a mouse, with a normal measure E as its active extender, we consider when $E \cap N||\alpha$ is on the N sequence (for $\alpha < (\mathrm{crit}(E)^+)^N$).

Steel showed that for $n \leq \omega$, M_n satisfies a variant of $\mathsf{V} = \mathsf{K}$ in the intervals between its Woodin cardinals. This gives that M_n satisfies $\mathsf{V} = \mathsf{HOD}$. (Of course, M_n isn't allowed to refer to $\mathbb{E}^{M_n}$ to achieve this feat!) A proof for the $n < \omega$ case can be seen in [17]. This method extends somewhat further into the mouse hierarchy, but just how far seems to be unknown. We show $\mathsf{V} = \mathsf{HOD}$ in certain mice by another (but related) method. A premouse is *self-iterable* if it satisfies "I am iterable". (This isn't intended to be precise - there are of course varying degrees of self-iterability.) In §5, we isolate a quality, *extender-fullness* (closely related to the reuslts of §3), that an iteration strategy might enjoy. We show that a premouse modelling ZFC and being sufficiently extender-full self-iterable can identify its own extender sequence. Indeed, it is the unique one that the premouse thinks is sufficiently extender-full iterable. Therefore $\mathsf{V} = \mathsf{HOD}$ in such a premouse. For $n \leq \omega$, the results of §3 will show that M_n's self-iteration strategy is extender-full, so we obtain a new proof of $\mathsf{V} = \mathsf{HOD}$ there. We then prove that various other mice also have (barely) enough extender-full self-iterability. It's critical, though, that the mouse M in question is *tame*, a fine-structural statement of "there is no κ that's strong past a Woodin". Our proof of $\mathsf{V} = \mathsf{HOD}$ works in particular below the strong cardinal of the least non-tame mouse. Our approach breaks down soon after non-tame mice are reached, because of a lack of self-iterability (see 5.16).

In §6 we look at homogeneously Suslin sets in mice. Using Kunen's analysis of measures in $L[U]$, Steel observed that all homogeneously Suslin sets in $L[U]$ are $\mathbf{\Pi}^1_1$ (so the two pointclasses coincide, by ([4], 32.1) or ([3], 33.30)). This was generalized by Schindler and Koepke in [18] (we give some details in §6). We show that in mice in the region of $0^\P$ or below, and modelling ZFC, all homogeneously Suslin sets are $\mathbf{\Pi}^1_1$. We also show that for $n < \omega$, in M_n, all homogeneously Suslin sets are $\mathbf{\Delta}^1_{n+1}$; in fact they are correctly so, in that the definition also yields a $\mathbf{\Delta}^1_{n+1}$ set in V. By Martin and Steel's results in [7], all $\mathbf{\Pi}^1_n$ sets are homogeneously Suslin in M_n. Therefore:

Corollary (6.4)**.** *In M_n, the weakly homogeneously Suslin sets of reals are precisely the $\mathbf{\Sigma}^1_{n+1}$ sets.*

We also show that the correctly $\mathbf{\Delta}^1_{n+1}$ sets of M_n are exactly the $\mathrm{Col}(\omega, \delta_0)$-universally Baire sets of M_n, where δ_0 is the least Woodin cardinal of M_n. The question of the precise extent of the homogeneously Suslin sets in the projective hiearchy of M_n remains unsolved. (§6 is fairly independent of the rest of this thesis. It uses the notion of finite support discussed in §4, but this is straightforward.)

Finally, in §7, we discuss and correct some problems in the copying construction of [11] (some details are supplied in [22]), and simultaneously prove that "freely dropping" iterability follows from normal iterability. Here the antagonist of the iterability game may enforce drops in model and degree at will. This fact is needed in proving some of the preceding theorems.

Conventions and Notation

Our notion of *premouse* is that of [22].

For ν a cardinal, $\mathcal{H}_\nu$ denotes the collection of sets of size hereditarily less than ν.

Whenever we refer to an ordering on $\mathrm{OR}^{<\omega}$, it is to the lexiographic order with larger ordinals considered first. For instance, $\{4, 12, 182\} < \{3, 15, 182\}$ and $\{6\} < \{4, 6\}$.

We discuss the use of the terms *pre-extender* and *extender*. Given a rudimentarily closed M, a *pre-extender over* M and an *extender over* M are as in [22]. Otherwise, we liberally use the term *extender* to mean "pre-extender over some M". An extender is *total* if its measures measure all sets in V. We sometimes emphasise that an extender need not be total by calling it *partial*. If E is a total extender, E can be *countably complete* or *wellfounded* as usual.

All extenders we use have support of the form $X \subseteq \mathrm{OR}$. Typically $X = \gamma \cup q$ for some ordinal γ and $q \in \mathrm{OR}^{<\omega}$. Suppose E is an extender. $\mathrm{tc}(E)$ denotes the trivial completion of E, ν_E denotes the natural length of E, and $\mathrm{lh}(E)$ denotes the length of $\mathrm{tc}(E)$ (see [22] for definitions). For $X \subseteq \nu_E$, $E \upharpoonright X$ is the sub-extender using only co-ordinates in X. If $\kappa = \mathrm{crit}(E)$, $E \subseteq M \times [\gamma]^{<\omega}$ and E's component measures are M-total, then we say E *measures exactly* $\mathcal{P}(\kappa) \cap M$. If also $\alpha \in M$ whenever there is a wellorder of κ of ordertype α in M, then $(\kappa^+)^E$ denotes $(\kappa^+)^M$. Given some N over which E is a pre-extender, i^N_E denotes an ultrapower embedding from N to $\mathrm{Ult}(N, E)$, or $\mathrm{Ult}_k(N, E)$, depending on context; i_E will be used when N is understood. The notation $\mathrm{Ult}_k(E, F)$ (for F another extender) and $\circ_k$ are introduced in 3.12.

For P a premouse, $\mathbb{E}^P$ denotes the extender sequence of P, not including any active extender. F^P denotes the active extender. $\mathbb{E}^P_+$ denotes $\mathbb{E}^P \frown F^P$. Let $\alpha \leq \mathrm{OR}^P$ be a limit ordinal. Let's define $P|\alpha$ and $P||\alpha$. $P|\alpha \trianglelefteq P$, and $\mathrm{OR}^{P|\alpha} = \alpha$. $P||\alpha$ is the passive premouse of height α agreeing with P strictly below α. If $\mathbb{E}$ is a good extender sequence, $\mathcal{J}^{\mathbb{E}}$ denotes the premouse constructed from $\mathbb{E}$. $\mathcal{J}^{\mathbb{E}}_\alpha$ denotes $\mathcal{J}^{\mathbb{E}}|(\alpha \cdot \alpha)$. (I.e. it is the α^{th} level in the $\mathcal{J}^{\mathbb{E}}$-hierarchy. We generally only use this $\mathcal{J}$-notation when a premouse needs to refer to its own levels.) If P is sound, $\mathcal{J}_1(P)$ denotes the premouse of height $\mathrm{OR}^P + \omega$ extending P. If P is active, μ_P and ν_P denote $\mathrm{crit}(F^P)$ and ν_{F^P} respectively.

For definability over premice, we use the $r\Sigma_n$ hierarchy as in [22]. Since $r\Sigma_1 = \Sigma_1$, we just write "Σ_1". For P a premouse, $X \subseteq P$ and $n \leq \omega$, $\mathrm{Def}^P_n(X)$ denotes the set of points in P definable with an $r\Sigma_n$-term from parameters in X. $\mathrm{Hull}^P_n(X)$ denotes the transitive collapse of $(\mathrm{Def}^P_n(X), \mathbb{E}^P \cap P, F^P \cap P)$ (where F^P is coded amenably as for a premouse).

Given an iteration tree $\mathcal{T}$, $\kappa^{\mathcal{T}}_\alpha = \mathrm{crit}(E^{\mathcal{T}}_\alpha)$, $\nu^{\mathcal{T}}_\alpha = \nu_{E^{\mathcal{T}}_\alpha}$ and $\mathrm{lh}^{\mathcal{T}}_\alpha = \mathrm{lh}(E^{\mathcal{T}}_\alpha)$. $(M^*)^{\mathcal{T}}_{\alpha+1}$ is the model to which $E^{\mathcal{T}}_\alpha$ applies after any drop in model, and

$$(i^*)^{\mathcal{T}}_{\alpha+1,\beta} : (M^*)^{\mathcal{T}}_{\alpha+1} \to M^{\mathcal{T}}_\beta$$

is the canonical embedding, if it exists. If $\mathcal{T}$ has a last model and there is no drop on $\mathcal{T}$'s main branch (from its root to its last model), then $i^{\mathcal{T}}$ denotes the corresponding embedding. When $\mathcal{T}$ is clear from context, we may drop the superscript in any of this notation.

Iterability for a phalanx is for iterations such that $\mathrm{lh}(E)$ is strictly above all exchange ordinals for each E used in the iteration.

KP^* is the theory KP + "There are unboundedly many α's such that $\mathcal{J}_\alpha^{\mathbb{E}} \models \mathsf{KP}$".

2 Extenders Strong Below a Cardinal

Suppose N is a fully iterable mouse modelling ZFC, and E is a total, wellfounded extender in N. We are interested in just how E was constructed from $\mathbb{E}^N$. In this section we'll show that if E is nice enough, things are simple as possible: E's trivial completion is $\mathbb{E}^N_\alpha$ for some α. In order to state the main theorem of this section (2.9) in full generality, we first need to discuss *Dodd soundness*. However, this definition isn't required for the statement or proof of the simpler corollary 2.10, which still carries much of the utility of the theorem, so the impatient reader might skip ahead to there.

Definition 2.1 (Generators)**.** Let E be a short extender with $\mathrm{crit}(E) = \kappa$. Suppose P is a premouse such that E measures exactly $\mathcal{P}(\kappa) \cap P$ and $X \subseteq \nu_E$. X *generates* α if $\alpha = [a, f]^P_E$ for some $f \in P$ and $a \in X^{<\omega}$. X *generates* E or *suffices as generators* for E if every $\alpha < \nu_E$ is generated by X. Let $\alpha < \nu_E$ and $t \in \nu_E^{<\omega}$. Then α is a *t-generator* of E iff α is not generated by $\alpha \cup t$. Note that an $\emptyset$-generator is just a generator (in the sense of [22]). E is *finitely generated* if some finite set generates E.

The following notion, due to Steel and Dodd, is taken from ([14], §3). Notation there is a little different.

Definition 2.2 (Dodd parameter and projectum)**.** Let E be a short extender. We define E's Dodd parameter $t_E = \{(t_E)_0, \ldots, (t_E)_{k-1}\}$ and Dodd projectum τ_E. Given $t_E \restriction i = \{(t_E)_0, \ldots, (t_E)_{i-1}\}$, $(t_E)_i$ is the largest $t_E\restriction i$-generator of E that is $\geq (\kappa^+)^E$. Notice $(t_E)_i < (t_E)_{i-1}$; $k = |t_E|$ is large as possible. τ_E is the sup of $(\kappa^+)^E$ and all t_E-generators of E.

The following is straightforward to prove (see ([14], §3) for some details).

Fact 2.3. *Let E be a short extender. Then τ_E is the least $\tau \geq (\kappa^+)^E$ such that there is $t \in \mathrm{OR}^{<\omega}$ with $\tau \cup t$ generating E. t_E is least in $\mathrm{OR}^{<\omega}$ witnessing this fact.*

Suppose that P is a premouse, $\kappa = \mathrm{crit}(E)$, E measures exactly $\mathcal{P}(\kappa) \cap P$, and

$$P|(\kappa^+)^P = \mathrm{Ult}(P, E)|(\kappa^+)^{\mathrm{Ult}(P,E)}.$$

If $\tau_E \in \mathrm{wfp}(\mathrm{Ult}(P, E))$ then τ_E is a cardinal of $\mathrm{Ult}(P, E)$. If $\tau_E = (\kappa^+)^P$ then E is generated by $t_E \cup \{\mathrm{crit}(E)\}$. If $\tau_E > (\kappa^+)^E$ then E is not finitely generated.

Remark 2.4. If $\tau_E \notin \mathrm{wfp}(\mathrm{Ult}(P, E))$ then $\tau_E = \nu_E$ is a limit of $\mathrm{Ult}(P, E)$-cardinals.

Definition 2.5. Let E, P be as in 2.1 and $t = t_E$. E is *Dodd-solid* iff for each $i < |t|$, $E \restriction (t_i \cup t \restriction i) \in \mathrm{Ult}(P, E)$. E is *Dodd-sound* iff it is Dodd-solid and, if $\tau_E > (\kappa^+)^E$ then for each $\alpha < \tau_E$, $E \restriction (\alpha \cup t_E) \in \mathrm{Ult}(P, E)$.

Remark 2.6. With E, P, κ as above, if $\tau_E = (\kappa^+)^E$, it may be the case that κ is a t_E-generator. In this case one might expect Dodd-solidity should require $E \restriction t_E \in \mathrm{Ult}(P, E)$ also. This, however, is not the standard definition. It may follow from the proof of 2.7, though we haven't investigated this.

The following critical fact is proven by ([14], 3.2) and a correction in ([20], 4.1).

Fact 2.7 (Steel)**.** *Let P be an active 1-sound, $(0, \omega_1, \omega_1 + 1)$-iterable premouse. Then F^P is Dodd-sound.*

Lemma 2.8. *Suppose N is a premouse modelling KP^* and "E a total (pre-)extender with critical point κ, and for each α, α^κ exists". Then $\mathrm{Ult}(N, E)$ is wellfounded iff E is countably complete in N. If this is so, then $\mathrm{Ult}(N, E)$ is a transitive class of N.*

Proof. Because N satisfies "α^κ exists for each α", the membership relation of $\mathrm{Ult}(N, E)$ is essentially set-like in N. Otherwise the argument is as for when $N \models \mathsf{ZFC}$. □(Lemma 2.8)

We're now ready to state the main theorem of this section.

Theorem 2.9. *Let N be an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse such that*

$$N \models \mathsf{KP}^* \ + \ E \textit{ is a total, short, countably complete, Dodd-sound extender, } \mathrm{crit}(E) = \kappa,$$

$$\tau = \tau_E \textit{ is a cardinal, } (\tau^\kappa)^+ \textit{ exists, and } \mathcal{H}_\tau \subseteq \mathrm{Ult}(L_{\kappa^+}[\mathbb{E}], E).$$

Then E is on $\mathbb{E}^N$.

Corollary 2.10 (Steel, Schlutzenberg)**.** *Let N be an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse such that*

$$N \models \mathsf{KP}^* \ + \ E \textit{ is a total, short, countably complete extender, } \mathrm{crit}(E) = \kappa,$$

$$\nu = \nu_E \textit{ is a cardinal, } (\nu^\kappa)^+ \textit{ exists, and } \mathcal{H}_\nu \subseteq \mathrm{Ult}(L_{\kappa^+}[\mathbb{E}], E).$$

Then E is on $\mathbb{E}^N$.

Remark 2.11. The basic idea behind the proof of 2.9 is like that of the initial segment condition in ([11], §10). Steel first proved 2.10 in the case that ν_E is regular in N and E coheres $\mathbb{E}^N$ below ν_E. The author then generalized this to obtain 2.9. The proof of 2.10 is the first half of the proof of 2.9, with $\tau = \nu$. This half does not involve the Dodd soundness of E.

Proof of Theorem 2.9. After some motivation, the proof will work through claims 1 to 6 below.

Suppose for the moment that E is type 3, and that $\mathrm{Ult}(N, E)$ is sufficiently iterable that we can successfully compare N with $\mathrm{Ult}(N, E)$. Suppose this results in iteration trees $\mathcal{T}$ on N and $\mathcal{S}$ on $\mathrm{Ult}(N, E)$, such that

- Both trees have the same final model Q,
- Neither tree drops on the branch leading to Q,
- The resulting embeddings commute; i.e., $i^{\mathcal{S}} \circ i_E = i^{\mathcal{T}}$,
- $\mathrm{crit}(i^{\mathcal{S}}) \geq \nu_E$.

Then by standard arguments, the first extender F used on $\mathcal{T}$'s main branch is compatible with E, and $\nu_F \geq \nu_E$, and it follows that E is on $\mathbb{E}^N$. We won't reach this directly, but first replace N with a hull M of $N|\lambda$ for some λ. With $\bar{E}$ the collapse of E, the countable completeness of E will guarantee the iterability of $\mathrm{Ult}(M, \bar{E})$ and (the M level versions of) the first three properties listed above. The fourth will require the iterability of the phalanx $(M, \mathrm{Ult}(M, \bar{E}), \nu_{\bar{E}})$. Most of the proof of the type 3 case is in establishing that iterability. In general, ν_E will be replaced with τ_E (these coincide when E is type 3).

We now drop the assumptions of the previous paragraph, and proceed with the proof. We first obtain the hull M.

We may assume that $\lambda = ((\tau^\kappa)^+)^N$ is the largest cardinal of N. $E \in N|\lambda$ and $\mathrm{lh}(E) < \lambda$, since E is coded by a subset of τ and computes a surjection of τ onto $\mathrm{lh}(E) = (\nu_E^+)^{\mathrm{Ult}(N|(\kappa^+)^N, E)}$. We may also assume E is the least counter-example to the theorem in the order of construction of N. Then E is actually definable over $N|\lambda$, since it's the least E' in $N|\lambda$ such that $N|\lambda$ satisfies "$E' \notin \mathbb{E}$, $\tau_{E'}^{\mathrm{crit}(E')}$ exists", and all first-order hypotheses of the theorem bar "$(\tau_{E'}^{\mathrm{crit}(E')})^+$ exists". Let

$$M = \mathrm{Hull}_\omega^{N|\lambda}(\emptyset)$$

and

$$\pi : M \to N|\lambda$$

be the hull embedding. So $E \in \mathrm{rg}(\pi)$.

By 2.8 and that $(\tau^\kappa)^+$ is the largest cardinal of N, $\mathrm{Ult}(N, E)$ is a (wellfounded) transitive class of N. Let η be the index of least difference between N and $\mathrm{Ult}(N, E)$. Let $\theta = \mathrm{card}^N(\eta)$. Let $\pi(\bar{E}) = E$, $\pi(\bar{\theta}) = \theta$, etc. The first claim will get us half way.

Claim 1. *The phalanx*

$$\mathcal{P} = (M, \mathrm{Ult}(M, \bar{E}), \bar{\theta})$$

is $\omega_1 + 1$ iterable.

Proof.

Case 1. θ is regular in N.

We want to obtain ordinals γ, ξ and embeddings

$$\psi : M \to N|\gamma, \quad \sigma : \mathrm{Ult}(M, \bar{E}) \to N|\xi,$$

such that

$$\psi \restriction \bar{\theta} = \sigma \restriction \bar{\theta}.$$

Using the freely dropping iterability of N, established in §7, these maps will allow us to copy an iteration tree on $(M, \mathrm{Ult}(M, \bar{E}), \bar{\theta})$ to a tree on N, completing the proof of the claim. We will in fact find such a $\psi, \sigma, \gamma, \xi$ inside N. The existence of such objects is a first order fact about N, and $i_E(M) = M$, so it suffices to show it is true in $\mathrm{Ult}(N, E)$ instead.

Let $\sigma^* = \pi \restriction \mathrm{Ult}(M, \bar{E})$. Note that

$$\sigma^* : \mathrm{Ult}(M, \bar{E}) \to \mathrm{Ult}(N|\lambda, E) = \mathrm{Ult}(N, E)|i_E(\lambda) = \mathrm{Ult}(N, E)|\lambda, \tag{1}$$

σ^* is elementary, and $\sigma^* \upharpoonright \bar{\theta} = \pi \upharpoonright \bar{\theta}$. (Applying the shift lemma also yields σ^*.) In fact there is σ' with these properties in $\mathrm{Ult}(N, E)$. For θ is regular in N and $\pi, M \in N$, so we have $\pi \upharpoonright \bar{\theta}$ is bounded in θ, and thus $\pi \upharpoonright \bar{\theta} \in N|\theta$. By agreement below θ, $\pi \upharpoonright \bar{\theta} \in \mathrm{Ult}(N, E)$. So $\mathrm{Ult}(N, E)$ has the (illfounded) tree searching for an embedding σ' with the properties above, and since $\mathrm{Ult}(N, E)$ is wellfounded and models KP^*, it has a branch. So we have

$$\mathrm{Ult}(N, E) \models \sigma' : \mathrm{Ult}(M, \bar{E}) \to \mathcal{J}^{\mathbb{E}}_{i_E(\lambda)} \ \&\ \sigma' \upharpoonright \bar{\theta} = \pi \upharpoonright \bar{\theta}.$$

We now want to convert π into an appropriate map $\psi : M \to N|\gamma$ for some $\gamma < \theta$. This will be done by taking some hulls of $N|\lambda$; for that we need some condensation facts.

Lemma 2.12. *Suppose P is an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse satisfying KP^*, δ is a cardinal of P and $\mathrm{cof}^P(\delta) > \omega$. Suppose $H \lhd P$, $\delta \leq \mathrm{OR}^H$ and H projects to δ. Then there are unboundedly many $\delta' < \delta$ such that*

$$\mathrm{Hull}^H_\omega(\delta') \trianglelefteq P.$$

Proof. Let $\beta < \delta$. Let $P_0 = \mathrm{Def}^P(\beta)$, and

$$P_{n+1} = \mathrm{Def}^P_\omega(P_n \cup \{P_n \cap \delta\}).$$

Let $\delta' = \sup_n(P_n \cap \delta)$ and $f : P' \to P$ be the uncollapse embedding of $\mathrm{Def}^P_\omega(\delta') = \bigcup_n P_n$. Since P sees this construction, $\delta' < \delta$. We claim δ' works. For $\mathrm{crit}(f) = \delta'$ and $f(\delta') = \delta$, so $\rho_\omega(P') = \delta'$. Thus by condensation ([22], 5.1), we either have $P' \trianglelefteq P$ or $P' \lhd \mathrm{Ult}(P|\delta', \mathbb{E}^P_{\delta'})$. In the latter case, δ' is a successor cardinal in the ultrapower, but the P_n's are constructed from P' in the same way they were from P, showing that $\mathrm{cof}^{\mathrm{Ult}}(\delta') = \omega$, a contradiction. □(Lemma 2.12)

Lemma 2.13. *Suppose P be an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse, not of the form $\mathcal{J}_1(P')$, $\delta < \zeta < \mathrm{OR}^P$, δ is a P-cardinal, $P|\zeta$ is passive and $P|\zeta \models \mathsf{ZF}^-$. Let $H = \mathrm{Hull}^{P|\zeta}_\omega(\delta)$. Then $H \lhd P$; moreover $\rho_1^{\mathcal{J}_1(H)} = \delta$, $p_1^{\mathcal{J}_1(H)} = \{\mathrm{OR}^H\}$, $\mathcal{J}_1(H)$ is ω-sound.*

Granting this lemma, we can establish the iterability of our phalanx (of Claim 1). Let $H = \mathrm{Hull}^{N|\lambda}_\omega(\theta)$. The lemmas give $\mathcal{J}_1(H) \trianglelefteq N$, and a $\theta' < \theta$ such that $\mathcal{J}_1(H') \trianglelefteq N$, where $\mathcal{J}_1(H') = \mathrm{Hull}^{\mathcal{J}_1(H)}_\omega(\theta')$ and $\mathrm{rg}(\pi) \cap \theta \subseteq \theta'$. There is a unique elementary $\psi' : M \to H' = N|\gamma$, and it agrees with π, and σ', below $\bar{\theta}$. Since $N|\theta = \mathrm{Ult}(N, E)|\theta$, we have $\psi' \in \mathrm{Ult}(N, E)$, so we're done.

Proof of Lemma 2.13. A strengthening of this lemma, in which $P = \mathcal{J}_1(P')$ is allowed, can be proven using the degree 1 version of condensation ([11], §8). However, the stated version is sufficient for our purposes, and we give a direct argument which involves a little less fine structure, and involves some arguments to be used later.

The failure of the lemma is a first order statement satisfied by P. So we may assume that $P = \mathrm{Hull}^P_\omega(\phi)$. Let $H = \mathrm{Hull}^{P|\zeta}(\delta)$.

Subclaim 1. $\rho_1^{\mathcal{J}_1(H)} = \delta$ and $\mathcal{J}_1(H)$ is ω-sound.

Proof. As $H = \mathrm{Hull}^H(\delta)$, we get $\rho_1^{\mathcal{J}_1(H)} \leq \delta$. (Note $\mathrm{Th}^H(\delta)$ is not in H and $H \models \mathsf{ZF}^-$, so it's not in $\mathcal{J}_1(H)$ either.) In fact $\rho_1^{\mathcal{J}_1(H)} = \delta$, as δ is a cardinal of P, $\mathcal{J}_1(H) \in P$, and P and $\mathcal{J}_1(H)$ agree below δ.

We claim that since $H \preccurlyeq_1 \mathcal{J}_1(H)$. This is because $H \models \mathsf{ZF}^-$. For if φ is pure Σ_1 ([11], §2), in the language of passive premice, let φ'_n, in the same language, be such that for any sound passive premouse B, and $x \in B$,

$$S_n(B) \models \varphi(x) \iff B \models \varphi'_n(x).$$

(Here $S_n(B)$ is the transitive structure n levels into the rud closure of $B \cup \{B\}$; $S_\omega(B) = \mathcal{J}_1(B)$.) Suppose $\mathcal{J}_1(H) \models \varphi(x)$ with $x \in H$. Then there's $n \in \omega$ such that $S_n(B) \models \varphi(x)$. Let $\theta \leq \mathrm{OR}^H$ be minimal such that $x \in H||\theta$ and $H||\theta \models \varphi'_n(x)$. Since $H \models \mathsf{ZF}^-$, $\theta < \mathrm{OR}^H$, and by minimality, $H||\theta \not\models \mathsf{ZF}^-$. Therefore $H|\theta$ is passive, so $H|(\theta+\omega) \models \varphi(x)$, so $H \models \varphi(x)$.

So the theories in question agree about pure Σ_1 formulae, which in fact implies they also agree about generalized Σ_1 formulae, as required.

So $\mathrm{Th}_1^{\mathcal{J}_1(H)}(\mathrm{OR}^H) \in \mathcal{J}_1(H)$. Therefore $p_1^{\mathcal{J}_1(H)} = \{\mathrm{OR}^H\}$ is 1-solid and since $H = \mathrm{Hull}_\omega^H(\delta)$, $\mathcal{J}_1(H)$ is 1-sound. Since $\rho_1^{\mathcal{J}_1(H)} = \delta$ is a cardinal of P, $\mathcal{J}_1(H)$ is ω-sound.

□(Subclaim 1)

Subclaim 2.

(a) For all $\beta \in H$, ${}^\beta H \cap (\mathcal{J}_1(H)) \subset H$;

(b) The phalanx

$$(P, \mathcal{J}_1(H), \delta)$$

is $\omega_1 + 1$-iterable.

Proof. (a) is because H models ZF^-.

For (b): The phalanx (P, H, δ) is iterable since the embeddings into P levels agree below the P cardinal δ. (So an iteration on this phalanx lifts to a freely droppping iteration of P, as discussed in §7.)

A normal degree ω tree $\mathcal{T}$ on H is essentially a normal degree 0 tree $\mathcal{T}'$ on $\mathcal{J}_1(H)$, using the same extenders. For $\alpha < \mathrm{lh}(\mathcal{T})$, if $[0,\alpha]_\mathcal{T}$ drops then $M_\alpha^{\mathcal{T}'} = M_\alpha^\mathcal{T}$, and otherwise $M_\alpha^{\mathcal{T}'} = \mathcal{J}_1(M_\alpha^\mathcal{T})$. By (a), one can inductively keep the association going. This extends easily to trees on $\mathcal{P}$ and $(P, \mathcal{J}_1(H), \delta)$. □(Subclaim 2)

Now compare P with $(P, \mathcal{J}_1(H), \delta)$, giving trees $\mathcal{T}$ and $\mathcal{U}$ respectively, with final models $Q^\mathcal{T}$ and $Q^\mathcal{U}$. As δ is a cardinal of P, no dropping occurs in $\mathcal{U}$ moving from the root P. The Closeness Lemma ([11], 6.1.5) shows that all other ultrapowers in $\mathcal{U}$ are close, so fine structure is preserved by the branch embeddings, as with normal iterations. Suppose $Q^\mathcal{U}$ is above P. Since P is pointwise definable (so $\mathcal{J}_1(P)$ projects to ω), it is straightforward to show $Q^\mathcal{T} = Q^\mathcal{U}$, $i^\mathcal{T}$ and $i^\mathcal{U}$ exist and $i^\mathcal{T} = i^\mathcal{U}$. This gives a contradiction via compatible extenders.

So $Q^\mathcal{U}$ is above $\mathcal{J}_1(H)$, and again since P is coded by $\mathrm{Th}(P)$, $Q^\mathcal{U} \trianglelefteq Q^\mathcal{T}$. So $\mathcal{U}$'s main branch doesn't drop, so $Q^\mathcal{U} = \mathcal{J}_1(H)$. If $Q^\mathcal{T} = \mathcal{J}_1(H)$, then again since $\mathrm{Th}(P) \notin P$ and

$\mathcal{J}_1(H) \in P$, there must be a drop from P to $Q^{\mathcal{T}}$, so $\mathcal{J}_1(H)$ is not sound, contradicting Subclaim 1. Since $\mathcal{J}_1(H)$ projects to δ,

$$\mathcal{J}_1(H) \lhd Q^{\mathcal{T}}|(\delta^+)^{Q^{\mathcal{T}}} = P||(\delta^+)^{Q^{\mathcal{T}}}.$$

□(Lemma 2.13)

As discussed after the statement of 2.13, this completes the proof of Claim 1 in Case 1.

Case 2. θ is singular in N.

This case works similarly; we just explain the differences. Let $\bar{\mu} = (\mathrm{cof}(\bar{\theta}))^M$ and $\langle \theta_\alpha \rangle_{\alpha<\bar{\mu}} \in M$ be an increasing sequence of successor cardinals of M converging to $\bar{\theta}$. We claim

$$\begin{gathered} \mathrm{Ult}(N, E) \models \exists \sigma, \gamma, \langle \pi_\alpha, \zeta_\alpha \rangle_{\alpha<\bar{\mu}}\,, \\ [\sigma : \mathrm{Ult}(M, \bar{E}) \to \mathcal{J}^{\mathbb{E}}_\gamma,\ \pi_\alpha : M \to \mathcal{J}^{\mathbb{E}}_{\zeta_\alpha}, \\ \pi_\alpha \restriction \theta_\alpha = \sigma \restriction \theta_\alpha]. \end{gathered} \tag{2}$$

From this we get that N has a similar σ and sequence of π_α's. Say $\kappa' < \bar{\theta}$. In iterating $\mathcal{P}$, we use σ as a copy map to lift $\mathrm{Ult}(M, \bar{E})$, and when using an extender with crit κ', we use π_α to lift M, where α is least such that $(\kappa'^+)^M \leq \theta_\alpha$.

To see (2), note (1) from the previous case is still true. However, if π is unbounded in θ, we may not actually have that $\pi \restriction \bar{\theta} \in \mathrm{Ult}(N, E)$. So we might not have a $\sigma' \in \mathrm{Ult}(N, E)$ agreeing with $\pi \restriction \bar{\theta}$. The existence of σ^* on the outside will be sufficient though. By the same argument as in Case 1, for each $\alpha < \bar{\mu}$ there is $\zeta_\alpha < \theta$ and $\pi'_\alpha \in N \cap \mathrm{Ult}(N, E)$, with $\pi'_\alpha : M \to N|\zeta'_\alpha = \mathrm{Ult}(N, E)|\zeta'_\alpha$, and $\pi'_\alpha \restriction \theta_\alpha = \pi \restriction \theta_\alpha = \sigma^* \restriction \theta_\alpha$. Now $\mathrm{Ult}(N, E)$ has the tree of attempts to simultaneously build a σ and sequence of π_α's with the desired properties. The existence of σ^* and the π'_α's shows this tree is illfounded, and therefore $\mathrm{Ult}(N, E)$ has a branch.

This completes the proof for Case 2. □(Claim 1)

Claim 2. *$M|\bar{\tau} = \mathrm{Ult}(M, \bar{E})|\bar{\tau}$ and $\bar{\theta} = \bar{\tau}$.*

Proof. Recall $\bar{\theta} = |\bar{\eta}|^M$, where $\bar{\eta}$ is the least disagreement between M and $\mathrm{Ult}(M, \bar{E})$. So it suffices to show $\bar{\theta} = \bar{\tau}$. Note $\bar{\theta} \leq \bar{\tau}$, since $\bar{E}$ is coded by a subset of $\bar{\tau}$ in M.

A comparison of M with $\mathcal{P} = (M, \mathrm{Ult}(M, \bar{E}), \bar{\theta})$ only uses extenders with index above $\bar{\theta}$. So by Claim 1, there is a successful one, producing trees $\mathcal{T}$ on M and $\mathcal{U}$ on $\mathcal{P}$. Let $U = \mathrm{Ult}(M, \bar{E})$. Similarly to the proof of Subclaim 2 in Lemma 2.13, since M is coded by $\mathrm{Th}(M) = \mathrm{Th}(U)$, the same final model Q is produced by $\mathcal{T}$ and $\mathcal{U}$, Q is above U in $\mathcal{U}$, and we have branch embeddings $i^{\mathcal{T}} : M \to Q$ and $i^{\mathcal{U}} : U \to Q$. $\bar{\theta} \leq \mathrm{crit}(i^{\mathcal{U}})$ by construction.

Assume $\bar{\theta} < \bar{\tau}$; then

$$\bar{\theta} < \bar{\eta} < (\bar{\theta}^+)^M \leq \bar{\tau}$$

as $\bar{\tau}$ is an M-cardinal. Now $\bar{\eta}$ is a cardinal of Q, as it indexes an extender used during the comparison. Also $(\bar{\theta}^+)^M = (\bar{\theta}^+)^U$ since $\mathcal{H}^M_{\bar{\tau}} \subseteq U$. So

$$\bar{\theta} \leq \mathrm{crit}(i^{\mathcal{U}}) < \bar{\eta} < (\bar{\theta}^+)^U.$$

But then the first extender $E^{\mathcal{U}}_\alpha$ hitting U on the main branch cannot be complete over U, because it only measures sets in $U|\bar{\eta}$. As $\mathcal{U}$'s main branch does not drop, this is a contradiction. □(Claim 2)

Notation. We proceed to show $\bar{E}$ is on $\mathbb{E}^M$, from the first order properties of M and the previous two claims. Since we will no longer refer directly to objects at the N level, we drop the bar notation.

The first paragraph of the proof of Claim 2 still holds, giving trees $\mathcal{T}$ on M and $\mathcal{U}$ on $\mathcal{P} = (M, U, \tau)$ with common last model Q. Let $i = i^{\mathcal{U}}$, $j = i^{\mathcal{T}}$. Then since M is pointwise definable, $i \circ i_E = j$. In the case E is type 1 or 3, so that $\tau = \nu_E$, we can now easily complete the proof. Since $\text{crit}(i^{\mathcal{U}}) \geq \nu_E$, we have the first extender $E^{\mathcal{T}}_\alpha$ used on the main branch of $\mathcal{T}$ is compatible with E. Since $\text{lh}(E^{\mathcal{T}}_0) > \nu_E$, $\text{lh}(E^{\mathcal{T}}_\alpha) > \nu_E$ also, so by the initial segment condition, E is on $\mathbb{E}^M$. This gives 2.10, but we haven't finished proving 2.9.

So assume E is type 2. Let Q be the last model of the comparison and b, c the main branches on U, M respectively. Let t be the Dodd parameter of E.

Claim 3.

(a) There is only one extender G used in c, which is type 2, with $\nu_G = i(\max t) + 1$.

(b) Suppose P is a model along b before Q. Then $i_{U,P}(\max t) \geq \text{crit}(i_{P,Q})$.

Proof. Toward (a), we first show that $i(\max t) + 1 \leq \nu_G$ by finding fragments of G in Q. By Dodd solidity of E,

$$W = E \restriction \max t \in U.$$

We claim that

$$E_j \restriction i(\max t) = i(W) \in Q.$$

To see this, consider W as the set of pairs

$$(A, i_E(A) \cap \max t)$$

such that $A \subseteq \text{crit}(E)$ and $A \in M$. Since $\text{crit}(E) < \text{crit}(i)$ and $j = i \circ i_E$, $i(W)$ is the set of pairs

$$(i(A), i(i_E(A) \cap \max t)) = (A, j(A) \cap i(\max t))$$

such that $A \cap \mathcal{P}(\text{crit} E) \cap M$, which is equivalent to $E_j \restriction i(\max t)$. But now if $\nu_G < i(\max t)+1$, then

$$G = E_j \restriction \nu_G = i(W) \restriction \nu_G \in Q.$$

On the other hand, $G \notin \text{Ult}(M, G)$, which contradicts the fact that Q and $\text{Ult}(M, G)$ have the same subsets of ν_G. So $i(\max t) + 1 \leq \nu_G$.

Now we prove $\nu_G \leq i(\max t) + 1$ and (b) together. If (b) holds let $P = Q$, and if (b) fails let P be the first counterexample to (b) along b. Either way, $i(\max t) = i_{U,P}(\max t)$.

Moreover, all generators of extenders used in $\mathcal{U}$ along the branch leading to P are below $i(\max t)$. Since $U = \text{Hull}^U(\max t + 1)$,

$$P = \text{Hull}^P(i(\max t) + 1).$$

Since $\text{crit}(i_{P,Q}) > i(\max t)$,

$$(i(\max t)^+)^Q \subseteq \text{Def}^Q(i(\max t) + 1).$$

Therefore no generators of $E_{i \circ i_E} = E_j$ lie between $i(\max t) + 1$ and $(i(\max t)^+)^Q$. So letting G be the first extender hitting M along c, G cannot have generators in that region either, so $\nu_G \leq i(\max t) + 1$. This gives (a).

Therefore

$$P = \text{Hull}^P(i(\max t) + 1) = \text{Hull}^Q(i(\max t) + 1) = \text{Hull}^{\text{Ult}(M,G)}(\nu_G) = \text{Ult}(M, G),$$

since $i^{\mathcal{U}}_{P,Q}$ and $i^{\mathcal{T}}_{\text{Ult}(M,G),Q}$ have critical points above $i(\max t) + 1 = \nu_G$. This gives $P = \text{Ult}(M, G)$. But these models appeared during a comparison, which implies they are the common final model, Q, giving (b). □(Claim 3)

We now know $i(t)$ is a set of generators of G. We need to investigate more carefully their roles in generating G. At this point, it's not clear that $i(t)$ is the Dodd parameter of G. We need to introduce a variant of the Dodd parameter and projectum, more analogous to the standard parameter and projectum. $i(t)$ is in fact this parameter. We'll also establish that iterations preserve this parameter and projectum nicely, which will allow us to trace the origins of G in $\mathcal{T}$.

Definition 2.14. Let $\pi : R \to S$ be a Σ_0-elementary embedding between premice. Suppose π is cardinal preserving and $\mu = \text{crit}(\pi)$ inaccessible in R. Suppose $E_\pi \notin S$, where E_π is the extender derived from π of length $\pi(\mu)$. The *Dodd-fragment parameter* of π, denoted $s_\pi = s = \{s_0, \ldots, s_{k-1}\} \in \text{OR}^{<\omega}$, is defined recursively as follows. Give $s \restriction i = \{s_0, \ldots, s_{i-1}\}$, s_i is the largest $\alpha \geq (\mu^+)^R$ such that

$$E_\pi \restriction (\alpha \cup s \restriction i) \in S,$$

if such exists. k is large as possible (note $s_{i+1} < s_i$). The *Dodd-fragment projectum* of π, denoted σ_π, is then the sup of $(\mu^+)^R$ and all α such that

$$E_\pi \restriction (\alpha \cup s_\pi) \in S.$$

(Note that since $\pi(\mu)$ is an S-cardinal, s_π and σ_π are in fact determined by $\pi(R|\mu)$.) One can also give a characterization as that of the Dodd parameter and projectum given in 2.3.

Given a premouse P active with extender E, the Dodd-fragment parameter and projectum of E (or of P) are the parameter and projectum of i^P_E.

We'll often refer to the Dodd-fragment parameter and projectum collectively as the *Dodd-fragment ordinals.*

Claim 4. *The Dodd-fragment ordinals of $i_G : M \to Q = \mathrm{Ult}(M, G)$ are $s_G = i(t_E) = i(t)$ and $\sigma_G = \tau_E = \tau$.*

Proof. As E is Dodd-sound, $s_E = t$ and $\sigma_E = \tau$. (Since E is coded by $E \restriction \tau \cup t$ and $E \notin U$, the Dodd-fragment ordinals can't be any higher.)

Suppose $E \restriction X \in U$. Then

$$i(E \restriction X) = G \restriction i(X).$$

This is just as for $i(E \restriction \max t) = G \restriction i(\max t)$, shown at the start of proving Claim 3 (note we now have $j = i_G$). So letting $X = (t_E)_k \cup t_E \restriction k$,

$$G \restriction [(i(t_E))_k \cup i(t_E) \restriction k] \in Q.$$

Similarly, if $\tau_E > (\mathrm{crit}(E)^+)^M$ and $\alpha < \tau_E$ then $G \restriction \alpha \cup i(t_E) \in Q$. Therefore, sufficient fragments of G with generators "below" $\tau_E \cup i(t_E)$ are in Q, to witness Claim 4 - we just need to see that $G \restriction \tau_E \cup i(t_E)$ is not in Q.

Suppose $s_G \restriction k = i(t_E) \restriction k$ but $(s_G)_k > i(t_E)_k$. Then

$$G \restriction [(i(t_E)_k + 1) \cup (i(t_E) \restriction k)] \in Q.$$

But this gives

$$E_{i \circ i_E} \restriction \tau_E \cup i(t_E) = G \restriction \tau_E \cup i(t_E) \in Q,$$

Since $\mathrm{crit}(i) \geq \tau_E$, this fragment of $E_{i \circ i_E}$ is isomorphic to $E \restriction \tau_E \cup t_E$, which fully determines E, and is coded as a subset of τ_E, But Q and U agree about such sets, so $E \in U$, a contradiction. This argument shows $s_G = i(t_E)$ and $\sigma_G = \tau_E$. $\square$(Claim 4)

The next claim will motivate the rest of the proof.

Claim 5.

(a) If G is Dodd-sound then $G = E$;

(b) If G is not the active extender of the model it is taken from, then $G = E$ is on $\mathbb{E}^M$.

Proof. By Dodd-soundness, $\sigma_G \cup s_G$ generates G. By Claim 4, commutativity and that $\tau_E \leq \mathrm{crit}(i)$,

$$E \restriction \tau_E \cup t_E \cong G \restriction \tau_E \cup i(t_E) = G \restriction \sigma_G \cup s_G = G.$$

This gives (a). For (b), G must be Dodd-sound by 2.7. Thus $G = E$, and $U = \mathrm{Ult}(M, E) = Q$, so there is no movement on the U side during the comparison. Recall η is the least disagreement between M and U. Since $\tau_G < \eta \leq \mathrm{lh}(G)$, η cannot be a cardinal in the model G comes from. Since η indexes the least disagreement, it must be that G is indexed at η in M. $\square$(Claim 5)

So we now assume that G is the active extender of the model R from which it is taken. Let R^* be $\mathfrak{C}_\omega(R)$. Notice the iteration from R^* to R drops immediately to degree 0 (thanks to Ralf Schindler for pointing out this simplification of our original argument). This is because $G \restriction \sigma_G \cup s_G$ is a $\mathbf{\Sigma}_1$ subset of R missing from R, so $\rho_1^R \leq \sigma_G = \tau_E$. But the comparison started above τ_E, so ultrapowers of degree ≥ 1 lead to models with first projectum strictly above τ_E.

Let G^* be R^*'s active extender. The following lemmas establish how the Dodd fragment ordinals of G are related to those of G^*. Note though that by 2.7, G^* is Dodd-sound. The first two lemmas are ([10], 2.1.4), which is based on ([11], 9.1).

Lemma 2.15. *Let P be an active premouse with $F = F^P$, and H a short extender over P with $\mathrm{crit}(H) < \nu_F$. Let $W = \mathrm{Ult}_0(P, H)$ and $i_H : P \to W$ be the canonical embedding. If $A_H \subseteq \mathrm{OR}$, $A_F \subseteq \nu_F$ suffice as generators for H and F respectively, then*

$$A_H \cup i_H``A_F$$

suffices as generators for F^W. (If H measures more sets than are in P, the hypothesis on A_H should be taken with respect to P, so $W = \{i_H^P(f)(a) \mid f \in P \ \&\ a \in A_H^{<\omega}\}$.)

Proof. Let $\alpha < \nu_{F^W}$; we want to generate α using ordinals in $A_H \cup i_H``A_F$. We have the ultrapower maps i_H, $i_F : P \to \mathrm{Ult}_0(P, F)$ and $i_{F^W} : W \to \mathrm{Ult}_0(W, F^W)$. Let $\psi : \mathrm{Ult}_0(P, F) \to \mathrm{Ult}_0(W, F^W)$ be given by the shift lemma. Then

$$\psi \circ i_F = i_{F^W} \circ i_H \ \ \& \ \ \psi \restriction \nu_F = i_H \restriction \nu_F. \tag{3}$$

Let $f \in P$ and $\bar{\beta} \in A_H^{<\omega}$ be such that $\alpha = i_H(f)(\bar{\beta})$. Let $g \in P$ and $\bar{\gamma} \in A_F^{<\omega}$ be such that $f = i_F(g)(\bar{\gamma})$. Let $\bar{\gamma}^* = i_H(\bar{\gamma})$. Then using (3),

$$\alpha = \psi(f)(\bar{\beta}) = \psi(i_F(g)(\bar{\gamma}))(\bar{\beta}) = (\psi \circ i_F)(g)(\bar{\gamma}^*)(\bar{\beta}) = i_{F^W}(i_H(g))(\bar{\gamma}^*)(\bar{\beta}).$$

□(Lemma 2.15)

Lemma 2.16. *Let P be a type 2 premouse with $F = F^P$ Dodd sound. Suppose H is a short extender over P with $\mathrm{crit}(H) < \tau_F$. Let $W = \mathrm{Ult}_0(P, H)$. Let $i_H : P \to W$ be the canonical embedding. Then F^W is Dodd sound, $t_{F^W} = i_H(t_F)$ and $\tau_{F^W} = \sup(i_H``\tau_F)$.*

Proof. By Lemma 2.15, we have

$$A = i_H(t_F) \cup \sup i_H``\tau_F$$

suffices as generators for F^W. Conversely, if F' is a fragment of F then $i_H(F')$ is a fragment of F^W (being a fragment is Π_1). Applying this to witnesses F' to the Dodd soundness of F, one sees that W has the desired fragments of F^W. □(Lemma 2.16)

Lemma 2.17. *Let P be a type 2 premouse and $F = F^P$. Suppose H is a short extender over P with $\sigma_F \leq \mathrm{crit}(H)$, such that P agrees with $W = \mathrm{Ult}_0(P, H)$ about $\mathcal{P}(\sigma_F)$. Let $i_H : P \to W$ be the canonical embedding. Then F^W is not Dodd sound; $s_{F^W} = i_H(s_F)$ and $\sigma_{F^W} = \sigma_F$.*

Proof. Again i_H maps fragments of F to fragments of F^W. If any stronger fragments of F^W are in W then in fact

$$F^W \restriction \sigma_F \cup i_H(s_F) \in W.$$

Now $\sigma_F \leq \operatorname{crit}(H)$ so this fragment is isomorphic to $F \restriction (\sigma_F \cup s_F)$, which isn't in P. Since P agrees with W about $\mathcal{P}(\sigma_F)$, it's not in W either.

Since $\operatorname{crit}(H)$ is an $i_H(s_F)$-generator with respect to F^W, F^W is not Dodd sound. (Note F and F^W have the same critical point, and P and W agree through its successor.) □(Lemma 2.17)

Now we return to the origins of G. Part (c) of the final claim completes the proof of the theorem.

Claim 6.

(a) $\operatorname{crit}(R^* \to R) \geq \tau_{G^*} = \sigma_G = \tau_E$*;*

(b) $R^* \trianglelefteq M$*;*

(c) $E = G^*$ *and* E *is on* $\mathbb{E}^M$*.*

Proof.

(a) Applying 2.16 and 2.17 to the (degree 0) branch leading from R^* to R, we get $\tau_{G^*} \leq \sigma_G$. Let $\zeta = \operatorname{crit}(R^* \to R)$. If $\zeta < \tau_{G^*}$, then

$$\sigma_G > i_{R^*,R}(\zeta) > \eta > \tau_E,$$

using the lemmas for the first inequality. This contradicts Claim 4. So $\operatorname{crit}(R^* \to R) \geq \tau_{G^*}$, so 2.17 gives $\tau_{G^*} = \sigma_G$.

(b) R^* is a proper segment of some model on the tree; $\rho_1^{R^*} \leq \tau_{G^*} < \eta$; η is a cardinal of all models on the tree other than M.

(c) By Claim 4, $i(t_E) = s_G$. By (a) and Lemma 2.17, $s_G = i_{R^*,R}(s_{G^*})$. So G^* is isomorphic to $G \restriction (\tau_{G^*} \cup s_G) = G \restriction (\tau_E \cup i(t_E))$, which is isomorphic to E. So by (b), E is on $\mathbb{E}^M$. □(Claim 6)(Theorem 2.9)

Corollary 2.18. *Let* $N \models \mathsf{ZFC}$ *be a mouse. Then*

(a) Every normal measure of N *is on* $\mathbb{E}^N$*.*

(b) If κ *is strong or Woodin in* N*, then it is so via extenders on* $\mathbb{E}^N$*.*

(c) If κ *is strong in* N*, then* $\mathbb{E}^N$ *is definable from* $N|\kappa$ *in the universe of* N*.*

(d) If τ *is an* N*-cardinal, and* P *is a premouse projecting to* τ*, extending* $N|\tau$*, and* N *has a total wellfounded short extender* E *such that* $P \trianglelefteq \operatorname{Ult}(N, E)$*, then* $P \trianglelefteq N$*.*

Proof. We leave (a)-(c) to the reader, and just prove (d). Assume N, E, P, τ are as there. Let M, $\bar{E}$, $\bar{\tau}$, etc., be defined as in the proof of the theorem. The iterability of the phalanx $(M, \mathrm{Ult}(M, \bar{E}), \bar{\tau})$ works as there. Then $\bar{P} \trianglelefteq Q$, since $\mathrm{crit}(i_{\bar{U},Q}) \geq \bar{\tau}$. But all extenders used in the comparison have index $> \bar{\tau}$, and $\bar{P}$ projects to $\bar{\tau}$, so $\bar{P} \trianglelefteq M$. □(Corollary 2.18)

Steel noticed that combining the last statement of 2.18 with an argument of Woodin's, we also get the following.

Corollary 2.19. *Let $N \models \mathsf{ZFC}$ be a mouse and θ be an uncountable cardinal in N. Then $N|(\theta^+)^N$ is definable from parameters over $\mathcal{H}^N_{(\theta^+)^N}$. Therefore $L(\mathcal{P}(\theta)^N) \models \mathsf{AC}$.*

Proof. Now we prove 2.19. There are two cases: first suppose that there is no cut-point γ of N such that $\theta \leq \gamma < (\theta^+)^N$. (Recall γ is a cut-point of N iff whenever E is on $\mathbb{E}^N_+$, if $\mathrm{crit}(E) < \gamma$ then $\mathrm{lh}(E) < \gamma$.) This implies that there are unboundedly many $\gamma < (\theta^+)^N$ indexing a total extender. So by 2.18(d), $\mathcal{H}^N_{(\theta^+)^N}$ can just look at total extenders E such that $N|\theta \trianglelefteq \mathrm{Ult}(N|\theta, E)$ and $\mathrm{Ult}(N|\theta, E)$ is wellfounded, to determine the levels of N projecting to θ. (If $\mathrm{Ult}(N|\theta, E)$ is wellfounded then E is countably complete in N.)

The proof of the second case is due to Woodin. Let $\tilde{M}$ be an enumeration of all countable elementary submodels of levels of N, that are members of N. $\tilde{M}$ is coded as a subset of ω_1^N in N. Let γ be a cut-point of N with $\theta \leq \gamma < (\theta^+)^N$. Let P be a sound premouse, $N|\gamma \trianglelefteq P$, P projecting to γ. Then we claim

$$N \models P \trianglelefteq \mathcal{J}[\mathbb{E}] \iff \text{every countable elementary submodel of } P \text{ is on } \tilde{M}.$$

If P's submodels are on the list, then let $P \in P' \trianglelefteq N$, with P' projecting to γ. In N, let $X \preccurlyeq P'$ be countable, with $P \in X$. The collapses $\bar{P}$ of P and $\bar{P}'$ of P' are on the list, so can be compared in V. Standard arguments show $\bar{P} \trianglelefteq \bar{P}'$, so $P \trianglelefteq P'$. Everything actually took place in $\mathcal{H}^N_{(\theta^+)^N}$, so we're done. □(Corollary 2.19)

Remark 2.20. The rest of this section contains a generalization of Lemma 2.16 to higher degrees, which we'll need in the next section.

Lemma 2.21. *Let P be a type 2 premouse, γ_P the index of the largest proper initial segment of $F = F^P$ on $\mathbb{E}^P$, $\kappa = \mathrm{crit}(F)$, and $X \subseteq \nu^P$. Suppose*

$$\kappa, \gamma_P \in \Lambda = \{x \in P \mid \exists f \in P, a \in X^{<\omega}\ (x = [a, f]^P_F)\}.$$

Then $\Lambda = \mathrm{Def}^P_1(X \cup (\kappa^+)^P)$.

Proof. Clearly $\Lambda \subseteq H = \mathrm{Hull}^P_1(X \cup (\kappa^+)^P)$ in general; we want to see $H \subseteq \Lambda$.

Let $F' = F{\restriction}X$ and

$$\tilde{\pi} : \mathrm{Ult}(P|(\kappa^+)^P, F') \to \mathrm{Ult}(P|(\kappa^+)^P, F)$$

be the canonical map. Because $\gamma_P \in \mathrm{rg}(\tilde{\pi})$, so is ν_F; in fact $\tilde{\pi}(\nu_{F'}) = \nu_F$. Let $R' = \mathrm{Ult}(P|(\kappa^+)^P, F')$ and

$$P' = (R'|(\nu^+_{F'})^{R'}, F')$$

(where F' is coded amenably as for a premouse). Let $\pi : P' \to P$ be the restriction of $\tilde{\pi}$. So $\Lambda = \mathrm{rg}(\pi)$. We claim π is Σ_1-elementary, which gives the lemma, since $\pi \restriction (\kappa^+)^P = \mathrm{id}$.

For this, note $\tilde{\pi} \circ i_{F'} = i_F$, which implies that for $A \in \mathcal{P}(\kappa) \cap P$ and $\gamma < \mathrm{OR}^{P'}$,

$$\pi(i_{F'}(A) \cap \gamma) = i_F(A) \cap \pi(\gamma).$$

Also $\pi(\nu_{F'}) = \nu_F$. So π respects the predicates for the active extenders of P and P', and $\pi(\gamma_{P'}) = \gamma_P$, where γ_P indexes the last proper segment of F' on $\mathbb{E}^{P'}$. Since also $\mathcal{P}(\kappa) \cap P \subseteq P'$, it follows that π is cofinal in P (see ([22], 2.9)). Therefore π is Σ_1-elementary. □(Lemma 2.21)

Lemma 2.22. *Let P be a k-sound, type 2, Dodd sound premouse, with $k \geq 1$. Let $\mu = \mathrm{crit}(F^P)$. Let H be a short extender over P with $\mathrm{crit}(H) < \rho_k^P$. If $R = \mathrm{Ult}_k(P, H)$ is wellfounded, then it is also Dodd sound. Moreover, $t_R = i_H(t_P)$; if $(\mu^+)^P = \tau_P$ then $\tau_R = i_H(\tau_P)$; if $(\mu^+)^P < \tau_P$ then $\tau_R = \sup i_H``\tau_P$.*

Proof. Of course if R is $(0, \omega_1, \omega_1 + 1)$-iterable, the lemma holds by 2.7. But we need to know it's true more generally, as we'll apply it where R is obtained as an iterate of P, via a strategy that, abstractly, we won't know is an $(\omega, \omega_1, \omega_1 + 1)$-strategy.

If $k > 1$, elementarity considerations easily imply R is Dodd sound.

Suppose $k = 1$. Let $F = F^P$, $\mu = \mathrm{crit}(F)$, $t = t_F$ and $\tau = \tau_F$. If $\tau = (\mu^+)^P$ then F is generated by $t \cup \{\mu\}$, a Π_2 condition, preserved by i_H. Also i_H maps fragments of F to fragments of F^R, so R is Dodd sound.

So assume $\tau_F > (\mathrm{crit}(F)^+)^P$. Then R has all fragments of F^R with generators "below" $i_H(t) \cup \sup i_H``\tau$. So it is enough to show this set generates $F^R$. Deny. Let $\kappa = \mathrm{crit}(H)$. Let $[a, f]$ represent an $i_H(t)$-generator for $F^R$ at least $\sup(i_H``\tau)$. Here $f : \kappa^n \to P$ is given by a $\Sigma_1^P(\{q\})$ term for some $q \in P$. Since the statement "α is an $i_H(t)$-generator" is Π_1, $f(u)$ must be a t-generator for measure one many u's. In particular, $\mathrm{rg}(f)$ includes a τ-cofinal set of t-generators.

Now 2.21 applies with $X = \lambda \cup t$, where $\lambda < \tau$ is large enough that $\mu, q, \gamma_P \in \Lambda$ (notation as in 2.21). But this contradicts the last sentence of the previous paragraph. □(Lemma 2.22)

Corollary 2.23. *If P is a type 2, Dodd sound premouse, then $\tau_P = \max((\mu_P^+)^P, \rho_1^P)$.*

Proof. This is like the proof of 2.21, but Dodd soundness is used to show the small ultrapowers, and so corresponding Σ_1 theories, are in P. □(Corollary 2.23)

3 Cohering Extenders

In this section we analyse the situation when an extender in a mouse "fits" on the mouse's sequence, or an iterate thereof. The first theorem stated is just a special case of 3.7 below.

Theorem 3.1. *Let N be an $(\omega, \omega_1, \omega_1{+}1)$-iterable mouse satisfying "there is no largest cardinal, and E is a wellfounded total short extender", and suppose $(N||\mathrm{lh}(E), E)$ is a premouse. Then E is on $\mathbb{E}^N$.*

Proof. See 3.7. □(Theorem 3.1)

The full theorem below assumes we have extender E fitting on $\mathbb{E}^P$ for some internal iterate P of N, and under enough hypotheses, shows that E is in fact on the P-sequence. An analogous theorem was proven by Schimmerling and Steel in ([16], §2) (using the theory of [21]), assuming there's no inner model with a Woodin, and a measurable exists. They showed that given a wellfounded iterate W of K, and an extender E which fits on the W sequence (in that $(W||\mathrm{lh}(E), E)$ is a premouse), such that E has sufficient weak background certificates, E was either used in the iteration or is on $\mathbb{E}^W_+$. They also proved another version dealing with sound mice projecting to ω (or at least below all critical points on the mouse's sequence) instead of K. Our theorem has a similar statement, but the iteration tree $\mathcal{T}$ and extender in question must be inside the mouse, and the requirement of background certificates is replaced by demanding the extender produce a wellfounded, class size ultrapower when used as a normal extension of $\mathcal{T}$.

Here is a related question.

Question 3.2 (Steel)**.** *Suppose $V = L[\mathbb{E}]$, and $L[\mathbb{E}]$ is fully iterable. Let E be an extender. Is E the extender of an iteration map?*

The theorem does deal in part with self-iterable mice, and we first discuss definability of iteration strategies a little.

Definition 3.3. An iteration tree $\mathcal{T}$ is *above* ρ if $\rho \leq \mathrm{crit}(E)$ for each extender E used in $\mathcal{T}$. Let P be a premouse. Σ is an η-iteration strategy for P *above* ρ if Σ works as an iteration strategy for normal trees on P, above ρ, of length $< \eta$.

Definition 3.4. A premouse P is *tame overlapping* α if whenever E is on $\mathbb{E}^P_+$ and $\mathrm{crit}(E) < \alpha \leq \delta < \mathrm{lh}(E)$,

$$P|\mathrm{lh}(E) \models \delta \text{ is not Woodin.}$$

A premouse is *tame* if it is tame overlapping α for every α.

Lemma 3.5. *Let P be an ω-sound premouse, with $\rho^P_\omega \leq \rho$, and suppose P is tame overlapping ρ. Then P has at most one $(\rho^+ + 1)$-iteration strategy above ρ.*

Proof. This generalizes the uniqueness of $(\omega_1 + 1)$-iteration strategies for ω-sound mice projecting to ω, so we just outline the differences. Suppose Σ, Γ are two strategies, $\mathcal{T}$ is above ρ, via both Σ and Γ, and $b = \Sigma(\mathcal{T}) \neq c = \Gamma(\mathcal{T})$. The least difference between $M^{\mathcal{T}}_b$ and $M^{\mathcal{T}}_c$

must occur within the Q-structure of the active side. Therefore one can compare the two sides, forming normal continuations $\mathcal{T}\,^\frown b\,^\frown \mathcal{U}_\Sigma$ and $\mathcal{T}\,^\frown c\,^\frown \mathcal{U}_\Gamma$: $\delta(\mathcal{T})$ remains Woodin up to the length of any extender used, so critical points never go below ρ. The remaining details are left to the reader. □(Lemma 3.5)

Now suppose W is a premouse modelling KP^* (or ZF^- suffices for our purposes), and either there's no largest cardinal in W, or the largest does not have measurable cofinality in W. Suppose $\Sigma \subseteq W$ is a set of pairs $(\mathcal{T}, b) \in W$, definable from parameters over W, and $\alpha \in W$. Then we claim that the statement

$$W \models \text{“}\Sigma \text{ is an } \alpha\text{-iteration strategy for the universe”}$$

makes sense. That is, there is a formula φ, with a predicate for Σ and constant for α, which asserts this, for premice of the form of W. We leave the justification of this to the reader in the case that there is no largest cardinal in W. Otherwise, let δ be the largest. In computing $j(\delta)$ for some non-dropping iteration map j, the hypothesis on δ means we need only consider functions bounded in δ. If $\delta < \gamma < \mu$ and $g : \delta \to \gamma$ is a bijection, then to compute $j(\gamma)$ we need consider only functions $g \circ f$, where f is bounded in δ. Thus the membership relation is set-like in W (note all the trees and branches being considered are themselves members of W). Since $W \models \mathsf{KP}^*$, wellfoundedness of iterates is therefore first-order. Moreover, if Σ is indeed an iteration strategy, then all models on trees via Σ are transitive classes of W.

For the theorem, we need to slightly strengthen the hypothesis that δ not have measurable cofinality in W.

Definition 3.6. Let N be a premouse, $\kappa \in N$. κ is *almost measurable* in N if $(\kappa^+)^N \in N$ and for unboundedly many $\alpha < (\kappa^+)^N$, $\mathbb{E}^N_\alpha$ has critical point κ.

Theorem 3.7. *Let N be a premouse satisfying KP^*. Suppose that in N, μ is regular and is not the successor of a singular cardinal with almost measurable cofinality. Suppose that $\mathcal{T} \in N|\mu$ is a normal iteration tree with final model P, and either*

(a) N is $(\omega, \omega_1, \omega_1 + 1)$-iterable and $\mathcal{T}$ is a finite tree on $N|\mu$, or

(b) $(\chi^+)^N < \mu$, and in $N|\mu$: Σ is a $(\chi^+ + 1)$-iteration strategy for the universe, definable from parameters, $\mathcal{T}$ is via Σ, and $\mathrm{lh}(\mathcal{T}) < (\chi^+)$, or

(c)

$$N \models \textit{I'm tame overlapping } \chi,\ \chi^+ < \mu,\ N|\mu \textit{ is } (\chi^+ + 1)\textit{-iterable via } \Sigma$$

$$\textit{and } \mathcal{T} \textit{ is based on } N|\chi^+,\ \textit{via } \Sigma,\ \textit{above } \chi.$$

In any case, suppose further that $E \in N$, $(P||\mathrm{lh}(E), E)$ is a premouse; $\sup_\beta \mathrm{lh}(E^{\mathcal{T}}_\beta) < \mathrm{lh}(E)$; ξ is the least ξ' such that $\mathrm{crit}(E) < \nu^{\mathcal{T}}_{\xi'}$; $i^{\mathcal{T}}_{0,\xi}$ exists; E measures all of $\mathcal{P}(\mathrm{crit}(E)) \cap M^{\mathcal{T}}_\xi$; $\mathrm{Ult}(M^{\mathcal{T}}_\xi, E)$ is wellfounded.

Then E is on $\mathbb{E}^P_+$.

Remark 3.8. One might formulate other suitable self-iterability hypotheses, though our proof depends on the definability of the strategy. We will use the following generalization in §5. 3.7 works just as well if Σ is only an iteration strategy above some cut-point η of N. We leave the generalization of the proof to the reader. We will also use a cross between (a) and (c) in §5.

Remark 3.9. There is a counterexample beyond a superstrong. Suppose M is a structure satisfying the premouse axioms, except for the requirement that no extender on $\mathbb{E}_+^M$ be of superstrong type. Let E be a total type 2 extender on M's sequence, and κ the largest cardinal of $\mathrm{Ult}(M,E)$ below $\mathrm{lh}(E)$. Suppose κ is superstrong in $\mathrm{Ult}(M,E)$, and F is an extender on $\mathbb{E}^{\mathrm{Ult}(M,E)}$ witnessing superstrongness. Note that F (normally) applies exactly to $M|\mathrm{lh}(E)$, with degree 0. But $\mathrm{Ult}(M|\mathrm{lh}(E),F)$ is an active premouse, distinct from $\mathrm{Ult}(M,E)|\mathrm{lh}(F)$, with the same reduct. (However, since both extenders are on the sequence of iterates that aren't far from being normal, this doesn't appear to be a strong failure. Moreover, in Jensen's λ-indexing, the extenders are not indexed at the same point.)

Proof of 3.7. The proof will take the remainder of this section. The overall approach is like that of 2.9, but the details differ. We'll first give a brief introduction (plenty of details are to follow). Assume for simplicity that E coheres with N (so $\mathcal{T} = \langle N\rangle$). As in 2.9, we'd like to compare $\mathrm{Ult}(N,E)$ with N, producing a common final model, commuting maps, and a critical point at least ν_E on the $\mathrm{Ult}(N,E)$ side. Again, we'll replace N with a pointwise definable M, and (with bar notation as usual), compare M with $(M,\mathrm{Ult}(M,\bar{E}),\bar{\theta})$. But to prove this phalanx is iterable (before knowing the conclusion of the theorem) the highest we can set $\bar{\theta}$ at is the largest cardinal of $\bar{P}||\mathrm{lh}(\bar{E})$. This has the undesired consequence that if $\bar{E}$ (or E) is type 2, so that $\bar{\theta} < \nu_{\bar{E}}$, then the iteration map on $\mathrm{Ult}(M,\bar{E})$ may move generators of $\bar{E}$. We dealt with this in 2.9 by analyzing how the witnesses to Dodd soundness move under iteration, on both sides of the comparison. Here things are more complicated, as E may well be Dodd unsound. However, by similar analysis, we can still match up the Dodd-fragment ordinals of $\bar{E}$ with those of $i_{M,Q}$. We'll see that the subextender $\bar{E}\restriction X$ they generate appears on a normal iterate of M, derived from the comparison tree on M. But then we can factor $i_{\bar{E}}$ and $i_{M,Q}$ through $\mathrm{Ult}(M,\bar{E}\restriction X)$, and the analysis can be repeated, with this ultrapower replacing M. Continuing in this way, through larger subextenders of $\bar{E}$, we will see that $\bar{E}$ itself appears on a normal iterate M' of M. But we're assuming (for this paragraph) that $\bar{E}$ actually coheres $\mathbb{E}^M$, and therefore $M' = M$, finishing the proof. We now drop the simplifying assumptions of this paragraph and commence with the details.

We may assume that μ is the largest cardinal in N.

Because $\mathrm{cof}^N(\delta)$ isn't almost measurable, the discussion preceding theorem statement shows that $\mathrm{Ult}(M^{\mathcal{T}}_\xi,E)$ is also a transitive class of $N|\mu$. (If $\mathrm{cof}^N(\delta)$ isn't measurable but *is* almost, then maybe E is type 1 and $\mathrm{crit}(E) = i^{\mathcal{T}}_{0,\xi}(\mathrm{cof}^N(\delta))$.) So the wellfoundedness of this model is a definable notion over $N|\mu$.

In case (b) or (c), we claim that in N, Σ is a $(0,\chi^+{+}1)$-iteration strategy on the universe, for $N|\mu$-based trees. For otherwise there is a non-dropping branch to an illfounded model P by Σ. Let j be the iteration map. Since μ is N's largest cardinal, $({}^{<\mu}\mu)^N \in N$ and $N \models \mathsf{KP}^*$, the membership relation of P is set-like in N, and N has sequences $\langle f_i \rangle \in N$, $\langle a_i \rangle \in N|\mu$ such that $j(f_{i+1})(a_{i+1}) \in j(f_i)(a_i)$. Since all critical points are $< \mu$ and μ is regular, $\langle f_i \rangle$ can be converted to $\langle \bar{f}_i \rangle \in N|\mu$ which still witness illfoundedness; contradiction.

In any case, (a), (b) or (c), let $\mathcal{T}'$ be the liftup of $\mathcal{T}$ to a degree-0 tree on N. As above, the models of $\mathcal{T}'$ and $\mathrm{Ult}(M_\xi^{\mathcal{T}'}, E)$ are wellfounded and are transitive classes of N.

In case (c), note that the restriction of Σ to a χ^+-iteration strategy above χ on $N|(\chi^+)^N$ is a point in $N|\mu$, definable over $N|\mu$ without parameters. Indeed, in $N|\mu$ it is the unique such strategy which extends to a $\chi^+ + 1$-strategy, in that each tree of length χ^+ has a cofinal (so wellfounded) branch. (Even if $\mu = (\chi^+ +)^N$, this is expressible over $N|\mu$.) By 3.5 applied in N, such a strategy must agree with Σ.

By the preceding discussion, being a counterexample to the theorem is first-order over $N|\mu$, so we may assume ours is the least such. Let $M = \mathrm{Hull}_\omega^{N|\mu}(\emptyset)$, $\pi : M \to N|\mu$, and let $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, etc. π provides liftup maps from $\Phi(\bar{\mathcal{T}})$ to $\Phi(\mathcal{T})$ (i.e. $\pi \restriction M_\gamma^{\bar{\mathcal{T}}}$ is the liftup from its domain to $M_{\pi(\gamma)}^{\mathcal{T}}$). (The reader can check that π's elementarity ensures that it lifts a normal continuation of $\Phi(\bar{\mathcal{T}})$ to a normal continuation of $\Phi(\mathcal{T})$.) So we know the $\Phi(\bar{\mathcal{T}})$ part of the claim to follow.

Let $\bar{\theta}$ be the largest cardinal of $\bar{P}||\mathrm{lh}(\bar{E})$ and $\pi(\bar{\theta}) = \theta$.

Claim 1. *The phalanxes* $\Phi(\bar{\mathcal{T}})$ *and*

$$(\Phi(\bar{\mathcal{T}}), \mathrm{Ult}(\bar{R}, \bar{E}), \bar{\theta})$$

are $\omega_1 + 1$*-iterable, in* V *for case (a), or in* N *for case (b) or (c). (A tree on the latter should return to the appropriate model of* $\Phi(\bar{\mathcal{T}})$ *for crits in below* $\bar{\theta}$*.)*

Proof. The iterability of the corresponding phalanx in 2.9 used heavily that E was strong below τ_E. Because we don't have this here, we need another approach. We'll retain π as our lifting map for the models on $\Phi(\bar{\mathcal{T}})$, and show that there's an embedding of $\mathrm{Ult}(\bar{R}, \bar{E})$ into a level of $\bar{P}$ agreeing with π below $\bar{\theta}$. We'll also discuss what happens when an extender returns to $\bar{P}$ or $\mathrm{Ult}(\bar{R}, \bar{E})$.

Let $R = M_\xi^{\mathcal{T}}$ and $R' = M_\xi^{\mathcal{T}'}$, the models E returns to in $\mathcal{T}$ and $\mathcal{T}'$; note $\mathrm{Ult}(R, E) = \mathrm{Ult}(R', E)|\mu$, and

$$\pi \restriction \mathrm{Ult}(\bar{R}, \bar{E}) : \mathrm{Ult}(\bar{R}, \bar{E}) \to \mathrm{Ult}(R, E).$$

Now by 2.13,

$$N \models \forall \lambda < \mu \text{ [if } \lambda \text{ is a cardinal then } \mathrm{Hull}_\omega^{N|\mu}(\lambda) \trianglelefteq \mathcal{J}^{\mathbb{E}} \text{]}.$$

$\mathrm{Ult}(R', E)$ satisfies the same statement, since the iteration maps fix μ and are at least Π_1 elementary. So letting

$$H' = \mathrm{Hull}_\omega^{\mathrm{Ult}(R,E)}(\mathrm{lh}(E)),$$

$\mathcal{J}_1(H')$ projects to $\mathrm{lh}(E)$, and $H' \triangleleft \mathrm{Ult}(R, E)$. $\mathrm{Ult}(R, E)$ satisfies Lemma 2.12 since N does, so we can get an $\gamma < \mathrm{lh}(E)$, with $\pi(\nu_{\bar{E}}) < \gamma$, and σ such that $\sigma : \mathcal{J}_1(H) \to \mathcal{J}_1(H')$ is the uncollapse map of $\mathrm{Hull}_\omega^{\mathcal{J}_1(H')}(\gamma)$, with $\mathrm{crit}(\sigma) = \gamma$, and $\mathcal{J}_1(H) \triangleleft \mathrm{Ult}(R, E)$. But $\mathrm{Ult}(R, E)$ and P agree below $\mathrm{lh}(E)$. So we get $\psi : \mathrm{Ult}(\bar{R}, \bar{E}) \to H \triangleleft P$, with $\psi \restriction \nu_{\bar{E}} + 1 = \pi \restriction \nu_{\bar{E}} + 1$.

(In fact if $R = N|\mu$, then $\mathrm{cof}^N(\mathrm{lh}(E)) = (\mathrm{crit}(E)^+)^N$, so then we could take $\pi``\mathrm{lh}(\bar{E}) \subseteq \gamma$, so that ψ agrees with π below $\mathrm{lh}(\bar{E})$, but this doesn't help in the end.)

Consider building a tree $\mathcal{U}$ on $(\Phi(\bar{\mathcal{T}}), \mathrm{Ult}(\bar{R}, \bar{E}), \bar{\theta})$ (so $\mathrm{lh}(E_0^{\mathcal{U}}) > \bar{\theta}$). By our hypothesis on $\mathrm{lh}(\bar{E})$, $\sup_\beta \mathrm{lh}(E_\beta^{\bar{\mathcal{T}}}) \leq \bar{\theta}$, so $\psi(E_0^{\mathcal{U}})$ is a suitable exit extender from P. Except for using an extender applying to $\bar{P}$ or $\mathrm{Ult}(\bar{R}, \bar{E})$ (discussed next), the remarks above discussing the iterability of $\Phi(\bar{\mathcal{T}})$ show the copying process works.

Suppose $F = E_\alpha^{\mathcal{U}}$ has crit η, where $\sup_\beta \nu_\beta^{\bar{\mathcal{T}}} \leq \eta < \bar{\theta}$, so that F applies to $\bar{P}$. Let $\mathcal{U}'$ be the liftup of $\mathcal{U}$, ψ_α be the α^{th} liftup map and $F' = \psi_\alpha(F)$. Since $\bar{\theta} < \mathrm{lh}(E_0^{\mathcal{U}})$ and $\bar{\theta}$ is a cardinal of $\mathrm{Ult}(\bar{R}, \bar{E})$, F measures exactly $\mathcal{P}(\eta) \cap \bar{P}|\bar{\theta} = \mathcal{P}(\eta) \cap \bar{P}|\mathrm{lh}(\bar{E})$. Since $\theta < \mathrm{lh}(E_0^{\mathcal{U}'}) < \mathrm{lh}(E)$, F' measures exactly $\mathcal{P}(\pi(\eta)) \cap P|\theta = \mathcal{P}(\pi(\eta)) \cap P|\mathrm{lh}(E)$. Since $\pi(\mathrm{lh}(\bar{E})) = \mathrm{lh}(E)$, π preserves the correct level to drop to. Moreover, π agrees with ψ, and therefore ψ_α, below $\bar{\theta}$, so certainly on the measured subsets of η. Therefore the shift lemma applies.

Suppose F applies to $\mathrm{Ult}(\bar{R}, \bar{E})$. If it causes a drop, clearly ψ preserves the level to drop to. If not, we simply lift to H, applying $N|\mu$'s freely dropping iterability (§7).

Thus the lifted tree is a (freely dropping) normal continuation of $\mathcal{T}$. □(Claim 1)

Remark 3.10. The reason we can't move the exchange ordinal above $\bar{\theta}$ is as follows. Suppose we do move it up. It may be that $\bar{P}|\mathrm{lh}(\bar{E})$ projects to $\bar{\theta}$. (In fact, the theorem implies that it does.) Suppose we want to apply an extender with crit $\bar{\theta}$. This extender must go back to $\bar{P}$, and it measures exactly $\bar{P}|\mathrm{lh}(\bar{E})$, so there is a drop to that level. ψ was arranged with its range bounded in $\mathrm{lh}(E)$ (which was needed to get $H \trianglelefteq P$), and

$$\theta = \psi(\bar{\theta}) < \psi(\mathrm{lh}(E_0^{\mathcal{U}})) < \mathrm{lh}(E).$$

Since θ is largest cardinal below $\mathrm{lh}(E)$ in $P||\mathrm{lh}(E)$, the least level projecting to θ after $\psi(\mathrm{lh}(E_0^{\mathcal{U}}))$ is strictly below $\mathrm{lh}(E)$. But $\pi(P|\mathrm{lh}(\bar{E})) = P|\mathrm{lh}(E)$, so π does not preserve the correct level to drop to. So the copying process breaks down. We get iterability with the exchange ordinal at $\bar{\theta}$, but at the cost of possibly moving some generators of $\bar{E}$ whilst iterating.

Also notice we used the fact that Σ is a full class strategy for $N|\mu$. Suppose instead $\Sigma \in N|\mu$ is an $N|\alpha$-based strategy for $N|\mu$, where $\alpha < \mu$. Then the above lifting doesn't work, as π can produce extenders with length $> \alpha$.

Notation. We proceed to show $\bar{E}$ is on $\mathbb{E}^{\bar{P}}$, from the first order properties of M and the phalanx iterability. Since we will no longer refer directly to objects at the N level, we drop the bar notation.

We expect the following statement to be extractable from the rest of the proof. It is a variation of ([21], 8.6), which is a related statement about K, although there, the exchange ordinal $\theta = \nu_E$ instead.

Conjecture 3.11. *Let M be an ω-sound mouse projecting to ω, and $\mathcal{T}$ a correct tree on M, with last model P. Suppose $(P||\mathrm{lh}(E), E)$ is a premouse, and* crit(E) *is such that E would apply normally to R, with degree k (if it were on the P-sequence). Let θ be the largest cardinal of $P||\mathrm{lh}(E)$. Then E is on $\mathbb{E}_+^P$ iff the phalanx $(\Phi(\mathcal{T}), \mathrm{Ult}_k(R, E), \theta)$ is ω_1+1-iterable.*

Let $U = \mathrm{Ult}(R, E)$. Since U agrees with P below $\mathrm{lh}(E) > \theta$, the iterability gives us successful comparison trees $\mathcal{U}$ on $(\Phi(\mathcal{T}), \mathrm{Ult}(R, E), \theta)$ and $\mathcal{V}$ on $\Phi(\mathcal{T})$. As in Lemma 2.13, since M is pointwise definable, the same model Q is produced on both sides. We claim that Q is above U in $\mathcal{U}$, and there is no dropping leading to Q on either tree. This follows by standard arguments using the hull property and that M is pointwise definable. For example, suppose Q is fully sound, so that there is no dropping leading to it. Suppose Q is not above U; let $M_\alpha^\mathcal{T}$ be the root of Q in $\mathcal{U}$. Let $j : M_\alpha^\mathcal{T} \to Q$ be the iteration map and let $E_\beta^\mathcal{U}$ be the first extender of j. Note that $\nu_F \geq \mathrm{lh}(E)$ for all F used in $\mathcal{U}$. Note $\mathrm{Ult}(M_\alpha^\mathcal{T}, E_\beta^\mathcal{U} \restriction \gamma) = \mathrm{Hull}_\omega^Q(\gamma)$. Thus the initial segment condition for $E_\beta^\mathcal{U}$ shows $\mathrm{crit}(j)$ has the Q-hull property (i.e. $\mathcal{P}(\mathrm{crit}(j))^Q \subseteq \mathrm{Hull}_\omega^Q(\mathrm{crit}(j))$), but that for $(\mathrm{crit}(j)^+)^Q \leq \alpha < \mathrm{lh}(E)$, α does not. This just depends on Q, and gives that $M_\alpha^\mathcal{T}$ is the root of Q in $\mathcal{V}$, and the embedding is the same on both sides. But then the corresponding first extenders were compatible, and used in a comparison; contradiction. Similar reasoning shows there is no dropping leading to Q, and that R is the root of Q in $\mathcal{V}$.

Claim 2. *There's only one extender G used on $\mathcal{V}$'s branch from R to Q and $\nu_G = i^\mathcal{U}(\gamma+1)$, where γ is the largest generator of E.*

Proof. This isn't quite as in 2.9 as we don't know E is Dodd sound (and it may not be), and there is a difficulty if E is just beyond a type Z segment.

Let G be the first extender used on the $\mathcal{V}$ branch from R to Q. Let $\sigma \in [U, Q]_\mathcal{U}$ be such that σ is least with $M_\sigma^\mathcal{U} = Q$ or $\mathrm{crit}(i_{\sigma,Q}^\mathcal{U}) > i_{0,\sigma}^\mathcal{U}(\gamma) = \gamma'$. Then $\nu_G \leq \gamma' + 1$: otherwise $G \restriction \gamma' + 1 \in M_\sigma^\mathcal{U}$, but $G \restriction \gamma' + 1$ is the length $\gamma' + 1$ extender derived from $i_{U,\sigma}^\mathcal{U} \circ i_E$ (as $i^\mathcal{U} \circ i_E = i_{R,Q}^\mathcal{V}$ and $\mathrm{crit}(i^\mathcal{U}) > \mathrm{crit}(E)$). So by definition of σ, $M_\sigma^\mathcal{U} = \mathrm{Ult}(R, G \restriction \gamma' + 1)$, and $G \restriction \gamma' + 1$ collapses the successor of γ' in $M_\sigma^\mathcal{U}$.

Now if $E \restriction \gamma \in \mathrm{Ult}(R, E)$, then $i_{R,\sigma}^\mathcal{U}(E \restriction \gamma) \in Q$ is compatible with G, so $\nu_G \geq \gamma' + 1$. Otherwise E has a last proper segment, $E \restriction \delta + 1$, and it is type Z. So $E \restriction \delta \in \mathrm{Ult}(R, E)$. Let $\sigma_1 \in [U, Q]_\mathcal{U}$ be such that σ_1 is least with $M_{\sigma_1}^\mathcal{U} = Q$ or $\mathrm{crit}(i_{\sigma_1,Q}^\mathcal{U}) > i_{R,\sigma_1}^\mathcal{U}(\delta) = \delta_1$. As in the previous paragraph $\nu_G \geq \delta_1 + 1$. Since $E \restriction \delta + 1$ is type Z, $\gamma = (\delta^+)^{\mathrm{Ult}(R, E \restriction \delta)}$, so $E \restriction \delta$ collapses γ to δ. So $\gamma_1 = i_{R,\sigma_1}^\mathcal{U}(\gamma)$ is collapsed below $\delta_1 + 1$ in $M_{\sigma_1}^\mathcal{U}$. Therefore if $M_{\sigma_1}^\mathcal{U} \neq Q$, then $\mathrm{crit}(i_{\sigma_1,Q}^\mathcal{U}) > \gamma_1$. So in fact $\sigma_1 = \sigma$ and $\gamma_1 = \gamma'$. Now

$$\mathrm{crit}(i_{\mathrm{Ult}(R,G),Q}^\mathcal{V}) \geq \nu_G \geq \delta_1 + 1.$$

But then $M_\sigma^\mathcal{U}$, Q and $\mathrm{Ult}(R, G)$ agree about $\mathcal{P}(\delta_1)$, so in fact $\mathrm{crit}(i_{\mathrm{Ult}(R,G),Q}^\mathcal{V}) > \gamma'$. So

$$M_\sigma^\mathcal{U} = \mathrm{Hull}_\omega^{M_\sigma^\mathcal{U}}(\gamma' + 1) = \mathrm{Hull}_\omega^Q(\gamma' + 1) = \mathrm{Hull}_\omega^{\mathrm{Ult}(R,G)}(\gamma' + 1) = \mathrm{Ult}(R, G).$$

So in fact $M_\sigma^\mathcal{U} = Q = \mathrm{Ult}(R, G)$. However, $\gamma' \notin \mathrm{Hull}_\omega^Q(\gamma')$ since γ is a generator of E (a fact coded in $\mathrm{Th}_\omega^{\mathrm{Ult}(R,E)}(\{\gamma\})$), so $\gamma' + 1 = \nu_G$. □(Claim 2)

We now begin to work on analysing the Dodd unsoundness of an extender F used in an iteration tree, looking at the Dodd-fragment ordinals of F, and of extenders that brought about F's Dodd unsoundness, and so on. The following elementary fact underlies this.

Definition 3.12. Let P be a premouse with $F = F^P$, and F' an extender over P. Then $F' \circ_k P$ denotes $\mathrm{Ult}_k(P, E)$; $F' \circ_k F$ or $\mathrm{Ult}_k(F, F')$ denotes $F^{\mathrm{Ult}_k(P,F')}$. In the absence of parentheses, we take association of $\circ_k$ to the right; i.e.

$$F' \circ_k F \circ_l Q = F' \circ_k (F \circ_l Q).$$

Lemma 3.13 (Associativity of Extenders)**.** *Let P_u and P_l (upper and lower) be active premice, $E_u = F^{P_u}$, $E_l = F^{P_l}$, such that $\mathrm{crit}(E_u) > \mathrm{crit}(E_l)$, and Q a premouse. Suppose E_u measures exactly $\mathcal{P}(\mathrm{crit}(E_u)) \cap P_l$ and $\mathrm{crit}(E_u) < \nu_{E_l}$, and likewise E_l with respect to Q (though Q may be passive), and $\mathrm{crit}(E_l) < \rho_k^Q$. Then*

$$(E_u \circ_0 E_l) \circ_k Q = E_u \circ_k (E_l \circ_k Q),$$

where $\circ_i$ denotes applying the extender on the left to the object on the right with degree i. Moreover, $i^Q_{E_u \circ_0 E_l} = i^U_{E_u} \circ i^Q_{E_l}$ (the degree k ultrapower embeddings).

Proof. Let $U = \mathrm{Ult}_k(Q, E_l)$, and $i^{P_l}_{E_u}$ and $i^U_{E_u}$ be the ultrapower embeddings associated to $\mathrm{Ult}_0(P_l, E_u)$ and $\mathrm{Ult}_k(U, E_u)$ respectively. Note $\mathrm{crit}(E_u) < \rho_k^U$, $\mathrm{lh}(E_l) = \mathrm{OR}^{P_l}$ is a cardinal in U, and $P_l||\mathrm{OR}^{P_l} = U|\mathrm{OR}^{P_l}$. So $\mathrm{Ult}_0(P_l, E_u)$ agrees below its height with $\mathrm{Ult}_k(U, E_u)$, and $i^{P_l}_{E_u} = i^U_{E_u} \restriction \mathrm{OR}^{P_l}$. As in 2.15, $\sup i_{E_u}``\nu_{E_l} = \nu_{E_u \circ E_l}$. Moreover, for $A \subseteq \mathrm{crit}(E_l)$ in P_l,

$$i_{E_u \circ E_l}(A) \cap \nu_{E_u \circ E_l} = i_{E_u}(i_{E_l}(A) \cap \nu_{E_l}) \cap \nu_{E_u \circ E_l} \tag{4}$$

by the definition of 0-ultrapower. (This associativity generates the associativity overall.)

Now we define a map from $\mathrm{Ult}_k(Q, E_u \circ E_l)$ to $\mathrm{Ult}_k(U, E_u)$. For τ_q a k-term defined with parameter $q \in Q$, $a \in \nu_{E_l}^{<\omega}$, and $b \in \nu_{E_u}^{<\omega}$, let

$$[\tau_q, i_{E_u}(a) \cup b]^Q_{E_u \circ E_l} \longmapsto [\tau'_{(i^Q_{E_l}(q), a)}, b]^{\mathrm{Ult}_k(Q,E_l)}_{E_u},$$

where τ' is naturally derived from τ by converting some arguments to parameters. (As in 2.15, $\nu_{E_u} \cup i_{E_u}``\nu_{E_l}$ suffices as generators for $E_u \circ E_l$.) Los' Theorem and (4) shows this is well-defined and Σ_k-elementary, and it's clearly surjective. This isomorphism commutes with the ultrapower embeddings, which gives $i^Q_{E_u \circ_0 E_l} = i^U_{E_u} \circ i^Q_{E_l}$. □(Lemma 3.13)

Corollary 3.14. *Let $P_1, \ldots, P_n$ be active premice, $E_i = F^{P_i}$, with $\mathrm{crit}(E_i) > \mathrm{crit}(E_{i+1})$, and Q be a premouse. Suppose E_i measures exactly $\mathcal{P}(\mathrm{crit}(E_i)) \cap P_{i+1}$ and likewise for E_n with respect to Q, and that $\mathrm{crit}(E_n) < \rho_k^Q$. Then, with $\circ = \circ_0$,*

$$((\ldots(E_1 \circ E_2) \circ \ldots) \circ E_n) \circ_k Q = E_1 \circ_k (\ldots \circ_k (E_n \circ_k Q));$$

$$i^Q_{((\ldots(E_1 \circ E_2) \circ \ldots) \circ E_n)} = i_{E_1} \circ \ldots \circ i^Q_{E_n}.$$

Definition 3.15 (Dodd core)**.** Let G be an extender. The *Dodd core* of G is

$$\mathfrak{C}_D(G) = G \restriction \sigma_G \cup s_G.$$

Remark 3.16. Let S be a premouse such that every extender on $\mathbb{E}_+^S$ is Dodd sound. Suppose $\mathcal{W}$ is a normal tree on S and G is on the $M_\alpha^{\mathcal{W}}$ sequence. Then if G is not Dodd sound, lemmas 2.16 to 2.22 show $\mathfrak{C}_D(G)$ is the active extender of $(M^*)_{\beta+1}^{\mathcal{W}}$, where β is the least β' such that $\beta'+1 \leq_W \alpha$ and $M_{\beta'+1}^{\mathcal{W}}$'s active extender is not Dodd sound. Equivalently, β is W−pred$(\beta'+1)$ for the least β' such that $\beta'+1 \leq_W \alpha$, there's no model drop from $(M^*)_{\beta'+1}^{\mathcal{W}}$ to $M_\alpha^{\mathcal{W}}$ and $\mathrm{crit}((i^*)_{\beta'+1,\alpha}^{\mathcal{W}}) \geq \tau_F$, where $F = F^{(M^*)_{\beta'+1}^{\mathcal{W}}}$.

Definition 3.17 (Core sequence)**.** Let P', Q' be premice, and $j : P' \to Q'$ be fully elementary. The *core sequence* of j is the sequence $\langle Q_\alpha, j_\alpha \rangle$ defined as follows. $Q_0 = P'$ and $j_0 = j$. Given $j_\alpha : Q_\alpha \to Q'$, if $j_\alpha = \mathrm{id}$ or the Dodd-fragment ordinals for j_α are not defined, we finish; otherwise let s, σ be the Dodd-fragment ordinals for j_α and $H_\alpha = \mathrm{rg}(j_\alpha)$. Let $Q_{\alpha+1} = \mathrm{Hull}_\omega^{Q'}(H_\alpha \cup s \cup \sigma)$ and $j_{\alpha+1} : Q_{\alpha+1} \to Q'$ the uncollapse map. Take the natural limits at limit ordinals. Since $\mathrm{crit}(j_\alpha) \in H_{\alpha+1}$, the process terminates.

Definition 3.18 (Damage)**.** Let S, $\mathcal{W}$ and G be as in 3.16. We define the *damage structure* of G in $\mathcal{W}$, denoted $\mathrm{dam}^{\mathcal{W}}(G)$, and the relation $<_{\mathrm{dam}}^{\mathcal{W}}$. If G is Dodd-sound, then $\mathrm{dam}^{\mathcal{W}}(G) = \emptyset$. Otherwise let G, $\mathfrak{C}_D(G)$ be on the sequences of $M_{\alpha_G}^{\mathcal{W}}$, $M_\alpha^{\mathcal{W}}$ respectively. Let $\mathrm{dam}^{\mathcal{W}}(G)$ be the sequence with domain

$$\{\beta \mid \beta+1 \in (\alpha, \alpha_G]_W\},$$

such that

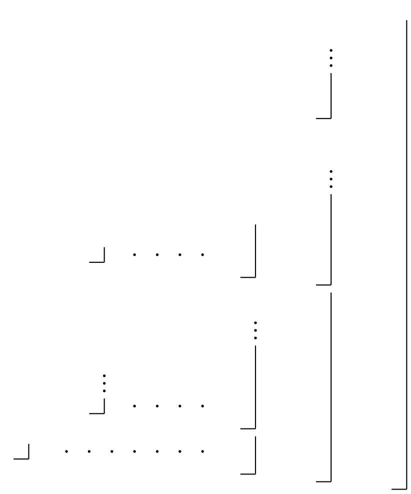
$$\mathrm{dam}^{\mathcal{W}}(G) : \beta \longmapsto (E_\beta^{\mathcal{W}}, \mathrm{dam}^{\mathcal{W}}(E_\beta^{\mathcal{W}})).$$

(Obviously there is a little superfluous information here.)

This is well-defined since $\mathrm{dom}(\mathrm{dam}^{\mathcal{W}}(G)) \subseteq \alpha_G$ - the deeper levels of the damage structure involve extenders used earlier in $\mathcal{W}$. Now let

$$\beta <_{\mathrm{dam}}^{\mathcal{W}} \gamma \iff E_\beta^{\mathcal{W}} <_{\mathrm{dam}}^{\mathcal{W}} E_\gamma^{\mathcal{W}} \iff E_\beta^{\mathcal{W}} \in \mathrm{tranclos}(\mathrm{dam}^{\mathcal{W}}(E_\gamma^{\mathcal{W}})).$$

Here is a diagram of a typical damage structure. An extender E is represented by the symbol $\rfloor$, with $\mathrm{crit}(E)$ and $\mathrm{lh}(E)$ corresponding to the bottom and top of the symbol respectively. $E <_{\mathrm{dam}} F$ iff E is pictured to the left of F. So Dodd sound extenders have no extenders to their left.



Remark 3.19. In the situation of 3.18, the remark above shows

$$\beta <_{\mathrm{dam}} \gamma \quad \Longrightarrow \quad \kappa_\gamma < \kappa_\beta < \mathrm{lh}_\beta < \mathrm{lh}_\gamma.$$

So if $\beta_1, \beta_2 \in \mathrm{dom}(\mathrm{dam}(E^{\mathcal{W}}_\gamma))$, with $\beta_1 < \beta_2$, then $\mathcal{W}$ has (finished damaging and) used $E^{\mathcal{W}}_{\beta_1}$, before using any of the extenders in the damage structure of $E^{\mathcal{W}}_{\beta_2}$. That is, if $\alpha_i <_{\mathrm{dam}} \beta_i$ $(i = 1, 2)$, then $\alpha_1 < \alpha_2$, since

$$\mathrm{lh}_{\alpha_1} < \mathrm{lh}_{\beta_1} \leq ((\kappa_{\beta_2})^+)^{E^{\mathcal{W}}_{\beta_2}} < \kappa_{\alpha_2} < \mathrm{lh}_{\alpha_2}.$$

(Though it may be that the tree has already dropped to $\mathfrak{C}_\omega(M^{\mathcal{W}}_{\beta_2})$ at some stage before applying $E^{\mathcal{W}}_{\beta_1}$.)

Notation. Let $\lambda \in \mathrm{OR}$ and F_α be an extender for $\alpha < \lambda$. Let P be a premouse. We define P_α for $\alpha \leq \lambda$. $P_0 = P$, $P_{\alpha+1} = \mathrm{Ult}_k(P_\alpha, F_\alpha)$, and take direct limits at limit α, as far as this definition makes sense. Then

$$_{\alpha<\lambda} \ldots \circ_k F_\alpha \circ_k \ldots \circ_k P$$

denotes P_λ. (So association is to the right, as above.) Similarly, if $F = F^P$,

$$_{\alpha<\lambda} \ldots \circ_k F_\alpha \circ_k \ldots \circ_k F$$

denotes F^{P_λ}. We will also make use of index sets $X \subseteq \mathrm{OR}$, replacing $\lambda \in \mathrm{OR}$.

Lemma 3.20. *Let M be a premouse with Dodd sound extenders on its sequence and $\mathcal{W}$ be a normal tree on M. Let $G = E^{\mathcal{W}}_{\alpha_G}$. Let $\langle F_\alpha \rangle_{0 \leq \alpha}$ enumerate*

$$\{\mathfrak{C}_D(E^{\mathcal{W}}_\beta) \mid \beta \leq_{\mathrm{dam}} \alpha_G\}$$

with increasing critical points. Let $G_0 = \mathrm{id}$, and

$$G_\alpha =_{1 \leq \gamma < \alpha} \ldots \circ F_\gamma \circ \ldots \circ F_0.$$

Let G_∞ be the limit of all G_α's.

The definition makes sense for each α; i.e. F_α measures precisely $\mathcal{P}(\mathrm{crit}(F_\alpha))^{G_\alpha}$ and the ultrapowers are wellfounded. Also $\rho_1^{G_\alpha} \leq \mathrm{crit}(F_\alpha)$.

Further, $G_\infty = G$, and for $1 \leq \alpha \leq \infty$ there's a normal tree $\mathcal{W}_\alpha$ on M and $\zeta \in \mathrm{OR}$ and $k \in \omega$ such that

- *G_α is on the sequence of $\mathcal{W}_\alpha$'s last model,*
- *$\mathcal{W}_\alpha \upharpoonright \zeta + 1 = \mathcal{W} \upharpoonright \zeta + 1$,*
- *$\mathrm{lh}(\mathcal{W}_\alpha) = \zeta + k + 1$,*
- *$\mathrm{crit}(G) < \nu^{\mathcal{W}}_\zeta$,*

- *if* $k > 0$ *then* $\mathrm{crit}(G) < \nu_\zeta^{\mathcal{W}_\alpha}$.

Finally, suppose P *is a* m*-sound premouse and* $\mathrm{crit}(G) < \rho_m^P$. *Then*

$$\mathrm{Ult}_m(P, G) =_{0\leq\gamma} \ldots \circ_m F_\gamma \circ_m \ldots \circ_m P,$$

and the associated embeddings (with domain P*) agree.*

Proof. Let $F_\alpha = \mathfrak{C}_D(E_{\alpha'}^{\mathcal{W}})$. $\mathcal{W}_\alpha$ is the part of $\mathcal{W}$ doing the damage preceding $E_{\alpha'}^{\mathcal{W}}$, then popping stack back out of the damage structure. Say $\alpha' \in \mathrm{dom}(\mathrm{dam}(E_{\beta'}^{\mathcal{W}}))$, and $\zeta = \mathcal{T}-\mathrm{pred}(\alpha'+1)$. If $G = E_{\beta'}^{\mathcal{W}}$, then we set $\mathcal{W}_\alpha = \mathcal{W}\restriction\zeta+1$.

Assume $E_{\beta'}^{\mathcal{W}} \neq G$. Still $\mathcal{W}_\alpha\restriction\zeta+1 = \mathcal{W}\restriction\zeta+1$, but here $k > 0$. Since $\alpha'+1 \leq_{\mathcal{W}} \beta'$ and there is no drop in $(\alpha'+1, \beta']_{\mathcal{T}}$ and $E_{\beta'}^{\mathcal{W}}$ isn't Dodd sound, there is a subextender I of $E_{\beta'}^{\mathcal{W}}$ on the sequence of $M_\zeta^{\mathcal{W}_\alpha}$. If α' is least in $\mathrm{dom}(\mathrm{dam}(E_{\beta'}^{\mathcal{W}}))$ then $I = \mathfrak{C}_D(E_{\beta'}^{\mathcal{W}})$ is active on $(M^*)_{\alpha'+1}^{\mathcal{W}} \trianglelefteq M_\zeta^{\mathcal{W}}$; otherwise I is Dodd unsound and is active on $M_\zeta^{\mathcal{W}}$. Set $E_\zeta^{\mathcal{W}_\alpha} = I$. (If I isn't active on $M_\zeta^{\mathcal{W}}$, still $\mathrm{lh}_\zeta^{\mathcal{W}} \leq \mathrm{lh}(I)$ since $E_{\alpha'}^{\mathcal{W}}$ triggers a drop to $M_\zeta^{\mathcal{W}}|\mathrm{lh}(I)$.) Say $\beta' \in \mathrm{dom}(\mathrm{dam}(E_{\gamma'}^{\mathcal{W}}))$. Then $E_{\beta'}^{\mathcal{W}}$ applies to a model on the branch leading to $M_{\gamma'}^{\mathcal{W}}$; moreover there is no drop in $(\beta'+1, \gamma']_{\mathcal{T}}$. Since $E_{\beta'}^{\mathcal{W}}$ and I have the same critical point and measure the same sets, and since $\mathrm{crit}(E_{\beta'}^{\mathcal{W}}) < \mathrm{crit}(E_{\alpha'}^{\mathcal{W}}) < \nu_\zeta^{\mathcal{W}}$, I also applies to the same (initial segment of the same) model. Moreover, $M_{\zeta+1}^{\mathcal{W}_\alpha}$ is active with a subextender of $E_{\gamma'}^{\mathcal{W}}$. If $G = E_{\gamma'}^{\mathcal{W}}$, $\mathcal{W}_\alpha$ is finished; otherwise set $E_{\zeta+1}^{\mathcal{W}_\alpha}$ to be $M_{\zeta+1}^{\mathcal{W}_\alpha}$'s active extender. Note it measures the same sets as $E_{\gamma'}^{\mathcal{W}}$. Continue in this way, always using active extenders, until exiting the damage structure (until applying an extender to a model on the branch leading to G). This must finish in finitely many steps.

We now prove inductively that G_α is on the sequence of the last model of $\mathcal{W}_\alpha$. For $\alpha = 1$, this was already observed. Assume it's true for some $\alpha \geq 1$; we consider $G_{\alpha+1}$.

This is an application of 3.13. Let ζ, k be as in the definition of $\mathcal{W}_\alpha$; other notation is also as there. Say H_0 is active on $(M^*)_{\alpha'+1}^{\mathcal{W}}$ and if $k > 0$, $H_0 = E_\zeta^{\mathcal{W}_\alpha}, \ldots, E_{\zeta+k-1}^{\mathcal{W}_\alpha}$ apply in $\mathcal{W}_\alpha$ to the extenders $H_1, \ldots, H_k$, respectively. (If $k > 0$, $H_0 = E_\zeta^{\mathcal{W}_\alpha}$.) So by induction and applying 3.14, the last model of W_α is active with

$$G_\alpha = ((H_0 \circ H_1) \circ H_2) \circ \ldots \circ H_k = H_0 \circ \ldots \circ H_k.$$

From the definition of $\mathcal{W}_{\alpha+1}$, the active extender of its last model is

$$((F_\alpha \circ H_0) \circ H_1) \circ \ldots \circ H_k = F_\alpha \circ G_\alpha = G_{\alpha+1}.$$

Moreover, with an appropriate m-sound premouse P, 3.13 gives $G_{\alpha+1} \circ_m P = F_\alpha \circ_m G_\alpha \circ_m P$, and that the ultrapower maps on P agree. By induction, this is equivalent to applying the F_β's, for $\beta \leq \alpha$, to P.

Let λ be a limit, and suppose we have the hypothesis below λ. Let $\beta \leq_{\mathrm{dam}} \alpha_G$ and F_λ be the Dodd core of $E_{\delta_0}^{\mathcal{W}}$, with $\delta_0 \in \mathrm{dom}(\mathrm{dam}(E_\beta^{\mathcal{W}}))$. Apply the following algorithm, as far as it works. Let δ_1 be largest in $\delta_0 \cap \mathrm{dom}(\mathrm{dam}(E_\beta^{\mathcal{W}}))$, and let δ_{i+2} be largest in $\mathrm{dom}(\mathrm{dam}(E_{\delta_{i+1}}^{\mathcal{W}}))$. (This process leads backwards in $\mathcal{W}$.) Let i be largest such that δ_i exists.

Note that δ_0 is not least in $\mathrm{dom}(\mathrm{dam}(E^{\mathcal{W}}_\beta))$, as otherwise it follows from remarks earlier that there is no $\kappa \in (\kappa^{\mathcal{W}}_\beta, \kappa^{\mathcal{W}}_{\delta_0})$ which is the crit of an extender in $\mathrm{dam}(G)$, contradicting $\kappa^{\mathcal{W}}_{\delta_0}$ being enumerated at a limit stage.

Assume $i = 0$ for now. Then $\zeta = \sup(\delta_0 \cap \mathrm{dom}(\mathrm{dam}(E^{\mathcal{W}}_\beta)))$ is a limit. It follows from 3.18 that $E^{\mathcal{W}}_{\delta_0}$ applies to $M^{\mathcal{W}}_\zeta$, a limit node in $\mathcal{W}$. Let H_0 be the active extender of $M^{\mathcal{W}}_\zeta$, and $H_1, \ldots, H_k$ be as for the successor case (again $k = 0$ is possible). We may assume inductively that the lemma applies to H_0, so letting F_α be the Dodd core of H_0 (equivalently of $E^{\mathcal{W}}_\beta$),

$$H_0 = {}_{\alpha<\gamma<\lambda} \ldots \circ F_\gamma \circ \ldots \circ F_\alpha,$$

The active extender of $M^{\mathcal{W}}_\lambda$'s final model is $(H_0 \circ H_1) \circ \ldots \circ H_k = H_0 \circ H_1 \circ \ldots \circ H_k$, which by the definition of $\mathcal{W}_\alpha$ and induction is $H_0 \circ G_\alpha$. Also inductively, the last statement of the lemma gives

$$H_0 \circ G_\alpha = {}_{\alpha\leq\gamma<\lambda} \ldots \circ F_\gamma \circ \ldots \circ G_\alpha = G_\lambda.$$

The "moreover" clause (for some m-sound P) also applies to H_0 and G_α, which gives it for G_λ. This finishes the $i = 0$ case. If $i > 0$ it's almost the same.

Now we show $G_\infty = G$. Clearly

$$G = {}_{\gamma'\in\mathrm{dom}(\mathrm{dam}(G))} \ldots \circ E^{\mathcal{T}}_{\gamma'} \circ \ldots \circ \mathfrak{C}_D(G), \tag{5}$$

Let γ', β' be successive elements of this set, or let γ' be the largest. Let $X = \{\alpha \mid \kappa^{\mathcal{T}}_{\gamma'} < \mathrm{crit}(F_\alpha) < \kappa^{\mathcal{T}}_{\beta'}\}$. By induction, for any premouse P,

$$\mathrm{Ult}_0(P, E^{\mathcal{T}}_{\gamma'}) = {}_{\alpha\in X} \ldots \circ F_\alpha \circ \ldots \circ \mathfrak{C}_D(E^{\mathcal{T}}_{\gamma'}) \circ P \tag{6}$$

and both embeddings on P agree. So for each γ' we can substitute a string of F_α's for $E^{\mathcal{T}}_{\gamma'}$ in (5), and associate to the right, to obtain

$$G = \ldots \circ F_\alpha \circ \ldots \circ F_0.$$

Finally, suppose P is m-sound and $\mathrm{crit}(G) < \rho^P_m$. We claim that

$$\mathrm{Ult}_m(P, G) = {}_{\gamma'\in\mathrm{dom}(\mathrm{dam}(G))} \ldots \circ_m E^{\mathcal{T}}_{\gamma'} \circ_m \ldots \circ_m \mathfrak{C}_D(G) \circ_m P, \tag{7}$$

and the corresponding embeddings agree. This is a straightforward extension of 3.13, and we leave the details to the reader. (Note that it may not make sense to associate *arbitrarily* here, as $\gamma' < \beta'$ might be successive elements of $\mathrm{dom}(\mathrm{dam}(G))$ with $\mathrm{lh}(E^{\mathcal{T}}_{\gamma'}) < \mathrm{crit}(E^{\mathcal{T}}_{\beta'})$. But it does make sense to associate to the right.) The degree-m version of (6) allows us to substitute a string of F_γ's for each $E^{\mathcal{T}}_{\gamma'}$, in (7), which finishes the proof. □(Lemma 3.20)

Let F_α, G_α and $\mathcal{T}_\alpha$ be as in 3.20 with respect to G and $\mathcal{T} {}^\frown \mathcal{V}$. By 3.20, the action of G on R, which results in the model Q, decomposes into the action of the F_α's. Let $j_{\alpha,Q}$ be the factor embedding from $Q_\alpha = \mathrm{Ult}(R, G_\alpha)$ to Q (given by the tail end of F_β's). We claim that $\langle Q_\alpha, j_{\alpha,Q}\rangle$ is the core sequence of $j : R \to Q$.

For this, we need to see the Dodd-fragment ordinals of $j_{\alpha,Q}$ are $s_{j_{\alpha,Q}} = j_{\alpha+1,Q}(t_{F_\alpha})$ and $\sigma_{j_{\alpha,Q}} = \tau_{F_\alpha}$. Since $\text{crit}(F_{\alpha+1}) \geq \tau_{F_\alpha}$, this is obvious if F_α is type 1 or 3, or if $\text{lh}(F_\alpha) < \text{crit}(F_{\alpha+1})$, so assume otherwise. Let $\alpha' \leq_{\text{dam}} G$ be such that $F_\alpha = \mathfrak{C}_D(E_{\alpha'}^{\mathcal{T}^\frown\mathcal{V}})$. Applying 3.20 to $E_{\alpha'}^{\mathcal{T}^\frown\mathcal{V}}$,

$$\text{Ult}(Q_\alpha, E_{\alpha'}^{\mathcal{T}^\frown\mathcal{V}}) = {}_{\gamma<_{\text{dam}}\alpha'} \ldots \circ F_\gamma \circ \ldots \circ F_\alpha \circ Q_\alpha,$$

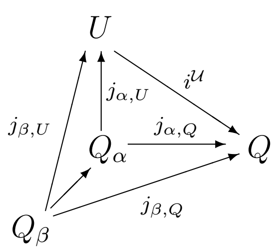
and the embeddings agree. If β is least such that $\beta \not<_{\text{dam}} \alpha$, then $\text{crit}(F_\beta) > \text{lh}(E_{\alpha'}^{\mathcal{T}^\frown\mathcal{V}})$, so $j_{\alpha,Q}$ is compatible with $E_{\alpha'}^{\mathcal{T}^\frown\mathcal{V}}$ through $\text{lh}(E_{\alpha'}^{\mathcal{T}^\frown\mathcal{V}})$. As F_α is Dodd-sound, applying the Dodd-fragment preservation of 2.17 to the iteration from F_α to $E_{\alpha'}^{\mathcal{T}^\frown\mathcal{V}}$ gives what we seek.

Remark 3.21. The compatibility of $j_{\alpha,Q}$ with $E_{\alpha'}^{\mathcal{T}}$ through $\nu_{\alpha'}^{\mathcal{T}}$ will be needed later.

Claim 3. *There is ε such that $Q_\varepsilon = \text{Ult}(R, E)$.*

Assuming the claim, we're essentially done: we know from 3.20 that $\mathcal{T}_\varepsilon$ has G_ε on the sequence of its final model, $\mathcal{T}_\varepsilon$ agrees with $\mathcal{T}^\frown\mathcal{V}$ through R and (since G applies to R in $\mathcal{V}$ and $\text{crit}(G) = \text{crit}(G_\varepsilon)$) G_ε applies to R normally. Since E did too, and $M = \text{Th}(M)$, we get $E = G_\varepsilon$. But then the agreement between $\mathcal{T}^\frown\mathcal{V}$ and $\mathcal{T}_\varepsilon$ (as in 3.20), and the coherence of E with P ($\mathcal{T}$'s last model) implies $\mathcal{T}_\varepsilon = \mathcal{T}$, so E is on the P-sequence, as desired.

Claim Proof. We will show inductively $j_{\alpha,Q}$ factors through $U = \text{Ult}(R, E)$, giving $j_{\alpha,U}$, such that the following diagram commutes:



Moreover, $\gamma \in \text{rg}(j_{1,U})$, where γ is the top generator of E and if $Q_\alpha \neq U$, then $\text{crit}(j_{\alpha,U}) < \theta$. The commutativity of the lower right triangle holds since the Q_α's form the core sequence of $j : R \to Q$.

Case 1. $\alpha = 0$.

Clearly this is true since $Q_0 = R = \text{Hull}_\omega^R(\text{crit}(G) \cap \text{crit}(E))$, and $\text{crit}(E) < \theta$ (so $\text{crit}(E) = \text{crit}(G)$).

Case 2. $\alpha = 1$.

Remember here E may be just beyond a type Z segment. Let γ be the top generator of E. Define $s \in \text{OR}^{<\omega}$ and $\sigma \in \text{OR}$ by $s_0 = \gamma$, then s_{i+1} is obtained from i_E and $s\restriction(i+1)$, and σ is obtained from i_E and s, as for the Dodd-fragment objects. Then $\sigma \leq \theta$. Otherwise it must be that $s < t_E$ in the lexiographic order, since $\tau_E \leq \theta$. Thus $\pi : \text{Hull}_\omega^{\text{Ult}(R,E)}(s \cup \sigma) \to \text{Ult}(R, E)$ is not the identity. So $\theta < \text{crit}(\pi) \leq \gamma$. But $\pi(\text{crit}(\pi))$ is a cardinal of $\text{Ult}(R, E)$, so is at least $\text{lh}(E)$, contradicting $\gamma \in s$.

Then $i^{\mathcal{U}}(s), \sigma$ are the Dodd-fragment objects s_G, σ_G for G. For Claim 2 showed $i^{\mathcal{U}}(\gamma)+1 = \nu_G$, so $(t_G)_0 = i^{\mathcal{U}}(\gamma)$. Also $\mathrm{crit}(i^{\mathcal{U}}) > \mathrm{crit}(E)$, and the iteration triangle commutes, so the fragments of E in $\mathrm{Ult}(R, E)$ are mapped to fragments of G in Q. So we just need the maximality of $i^{\mathcal{U}}(s), \sigma$ with respect to i_G. If they weren't maximal, then $G\restriction(\sigma \cup i^{\mathcal{U}}(s)) \in Q$. But this is isomorphic to $E\restriction(\sigma \cup s)$, and is coded as a subset of $\sigma \leq \theta \leq \mathrm{crit}(i^{\mathcal{U}})$.

So $t_G \cup \sigma_G \subseteq \mathrm{rg}(i^{\mathcal{U}})$, which gives the factorization of $j_{1,Q}$. We also saw $\gamma \in \mathrm{rg}(j_{1,Q})$.

Case 3. $\alpha = \beta + 1 > 1$.

Suppose the factoring holds at β, and $Q_\beta \neq U$. So $\kappa = \mathrm{crit}(j_{\beta,U}) \leq \gamma$. So in fact $j_{\beta,U}(\kappa) \leq \theta$: otherwise, as in Case 2, $j_{\beta,U}(\kappa) \geq \mathrm{lh}(E)$, contradicting $\gamma \in \mathrm{rg}(j_{1,U}) \subseteq \mathrm{rg}(j_{\beta,U})$.

Suppose at first that $j_{\beta,U}(\kappa) < \mathrm{crit}(i^{\mathcal{U}})$. Then $j_{\beta,U}(\kappa) = j_{\beta,Q}(\kappa)$. But $j_{\beta,Q} = j_{\beta+1,Q} \circ i_{F_\beta}$. The fragments of i_{F_β} in $\mathrm{Ult}(Q_\beta, F_\beta)$ are all below $i_{F_\beta}(\kappa)$, and $\sigma_{F_\beta} = \tau_{F_\beta} \leq \mathrm{crit}(j_{\beta+1,Q})$, so $j_{\beta+1,Q}$ maps maximal fragments of i_{F_β} to those of $j_{\beta,Q}$. So letting $s = j_{\beta+1,Q}(t), \sigma$ be the Dodd-fragment ordinals of $j_{\beta,Q}$, we have $s \cup \sigma \subseteq \theta \subseteq \mathrm{rg}(i^{\mathcal{U}})$. Since $Q_{\beta+1}$ is generated by $\mathrm{rg}(j_{\beta,\beta+1}) \cup t \cup \sigma$, we get $j_{\beta+1,Q}$ factors through U.

Now suppose $j_{\beta,U}(\kappa) = \theta = \mathrm{crit}(i^{\mathcal{U}})$. Let F be the extender of length θ derived from $j_{\beta,U}$. It can't be that $F \in \mathrm{Ult}(R, E)$ since otherwise $i^{\mathcal{U}}(F) \in Q$ is the extender of length $j_{\beta,Q}(\kappa)$ derived from $j_{\beta,Q}$ - this contradicts $j_{\beta,Q} = j_{\beta+1,Q} \circ i_{F_\beta}$, as in the previous paragraph. So we again get $s \cup \sigma \subseteq \theta$, and the factoring.

Case 4. α is a limit.

Since the Q_α's are the core sequence, the commutativity of the maps before stage α makes this case is easy.

This completes the proof of factoring. Since the Q_α's eventually reach Q, it must be that there is some stage ε with $Q_\varepsilon = U$. $\square$(Claim 3)(Theorem 3.7)

4 Measures and Partial Measures

Consider a mouse N satisfying "μ is a countably complete measure over some set" (plus say ZFC). 2.10 and 3.7 give different criteria which guarantee μ is on $\mathbb{E}^N$; e.g. normality is enough. We will now show that in general, i_μ is precisely the embedding of a finite iteration tree on N. This generalizes Kunen's result on the model $L[U]$ for one measurable, that all its measures are finite products of its unique normal measure.

Finite Support of an Iteration Tree

Toward our goal, we need to be able to capture a given element of a normal iterate with a finite normal iteration. That is, we want a finite iteration with liftup maps to the original one, with the given element in the range of the ultimate liftup map. The method is straightforward: find a subset of the tree sufficient to generate the given element, then perform a reverse copying construction to produce the finite tree. One must be a little careful, though, to ensure the resulting tree is normal. This tool is also used in §6.

Definition 4.1 (Finite Support)**.** Let $\mathcal{T}$ be a normal iteration tree on a premouse M of length $\theta + 1$, and let $B \subseteq M_\theta$ be a finite set. A hereditarily finite set A *supports* B (relative to $\mathcal{T}$) if the following properties hold.

Let $M_\alpha = M_\alpha^{\mathcal{T}}$. Then $A \subseteq \{(\alpha, x) \mid \alpha \in \theta + 1 \ \&\ x \in M_\alpha\}$. Let $(A)_\alpha$ denote the section of A at α. Let $S \subseteq \theta + 1$ be the left projection of A. Then $\theta \in S$ and $B \subseteq (A)_\theta$. Let $\alpha \in S$, $\alpha > 0$.

Case 1. $\alpha = \beta + 1$.

Let $\gamma = \mathcal{T}-\mathrm{pred}(\alpha)$. Then $\beta, \gamma, \gamma + 1 \in S$. (Note $\beta, \gamma + 1 \leq \alpha$.)

For x such that $(\alpha, x) \in A$, there are a_x, q_x such that $x = [a_x, f_{\tau_x, q_x}]^{(M^*)_\alpha}_{E^{\mathcal{T}}_\beta}$, and $(\beta, a_x), (\gamma, q_x) \in A$. (Here f_{τ_x,q_x} is the function given by the Skolem term τ_x and parameter $q_x \in (M^*)_\alpha$.)

If $E^{\mathcal{T}}_\beta \in \mathfrak{C}_0(M_\beta)$, then $(\beta, \mathrm{lh}(E^{\mathcal{T}}_\beta)) \in A$. Suppose M_β is active type 3, and $F = F^{M_\beta}$ is its active extender. If $E^{\mathcal{T}}_\beta = F$, there are a_ν, f_ν such that $\nu_F = [a_\nu, f_\nu]^{M_\beta}_F$ and $(\beta, a_\nu), (\beta, f_\nu) \in A$. If $\nu_F < \mathrm{lh}(E^{\mathcal{T}}_\beta) < \mathrm{OR}^{M_\beta}$ (so $E^{\mathcal{T}}_\beta \notin \mathfrak{C}_0(M_\beta)$), there are $a_{\mathrm{lh}}, f_{\mathrm{lh}}$ such that $\mathrm{lh}(E^{\mathcal{T}}_\beta) = [a_{\mathrm{lh}}, f_{\mathrm{lh}}]^{M_\beta}_F$ and $(\beta, a_{\mathrm{lh}}), (\beta, f_{\mathrm{lh}}) \in A$.

Case 2. α is a limit ordinal.

Then $S \cap \alpha \neq \emptyset$; let $\beta = \sup(S \cap \alpha)$. Then there is β' such that

- $0 <_{\mathcal{T}} \beta' <_{\mathcal{T}} \beta <_{\mathcal{T}} \alpha$
- $\beta' = \mathcal{T}-\mathrm{pred}(\beta)$ (so β is a successor)
- $i_{\beta',\alpha}$ exists and $\deg^{\mathcal{T}}(\beta') = \deg^{\mathcal{T}}(\alpha)$
- $(A)_\alpha \subseteq \mathrm{rg}(i_{\beta',\alpha})$

Moreover, $A \supseteq \{(\beta, x) \mid i_{\beta,\alpha}(x) \in (A)_\alpha\}$. It might seem that β' suffices as $\sup(S \cap \alpha)$, but choosing β instead of β' is needed to ensure normality of the finite tree.

This completes the definition of support. We now give an algorithm that passes from $\mathcal{T}, B$ as above to a support A for B. We'll recursively define sets S_i, A_i approximating the desired S, A, with $S_i \subseteq S_{i+1}$ and $A_i \subseteq A_{i+1}$. We'll also define ordinals $\alpha_i \in S_i$. $S_0 = \{\theta\}$, $A_0 = \{\theta\} \times B$ and $\alpha_0 = \theta$. Given α_n, S_n, A_n, if $\alpha_n > 0$, we process α_n, ensuring S_{n+1}, A_{n+1} satisfy the requirements of 4.1 for $\alpha = \alpha_n$. If α_n is a successor, let $S_{n+1} = S_n \cup \{\beta, \gamma, \gamma+1\}$ (notation as in 4.1) and enlarge A_n to A_{n+1} by adding the appropriate (β, a_x), (γ, q_x), etc., and also if $\gamma + 1 < \alpha_n$, adding $(\gamma+1, 0)$. If α_n is a limit, we can find $\beta > \beta' > \sup(S_n \cap \alpha)$ minimal with the required properties, and set $S_{n+1} = S_n \cup \{\beta\}$. Define A_{n+1} by adding the appropriate $i_{\beta,\alpha}$ preimages to A_n. Finally, $\alpha_{n+1} = \sup(S_{n+1} \cap \alpha_n)$; notice this exists by construction. The algorithm can be made definite by minimizing in some way to make choices.

Since $\alpha_{i+1} < \alpha_i$, there's n with $\alpha_n = 0$. Fixing this n, note that $\{\alpha_0, \ldots, \alpha_n\} = S_n$, so all elements of $S_n - \{0\}$ got processed at some stage in the construction. Notice that for $i \leq n$, $(A_i)_{\alpha_i} = (A_n)_{\alpha_i}$, so setting $A = \bigcup_{i \leq n} A_i$, it's easy to see that A supports B (and has projection $S = \bigcup_{i \leq n} S_i$).

Lemma 4.2. *Let $\mathcal{T}$ be a normal iteration tree on a premouse M, $\mathrm{lh}(\mathcal{T}) = \theta+1$, and $B \subseteq M^{\mathcal{T}}_\theta$ finite. There is a normal tree $\mathcal{U}$ on M with $\mathrm{lh}(\mathcal{U}) = n+1 < \omega$, with $\deg^{\mathcal{U}}(n) = \deg^{\mathcal{T}}(\theta)$, and a near $\deg^{\mathcal{T}}(\theta)$-embedding $\pi_n : M^{\mathcal{U}}_n \to M^{\mathcal{T}}_\theta$, with $B \subseteq \mathrm{rg}(\pi_n)$. Moreover, if $\mathcal{T}$'s main branch does not drop then neither does $\mathcal{U}$'s, and the main embeddings commute: $\pi_n \circ i^{\mathcal{U}} = i^{\mathcal{T}}$.*

Proof. Let A support B relative to $\mathcal{T}$. We will perform a "reverse copying construction", just copying down the parts of the tree appearing in A. The natural indexing set for $\mathcal{U}$ is S instead of an ordinal. Let the tree order $U = T \restriction S$ and drop/degree structure $D^{\mathcal{U}} = D^{\mathcal{T}} \restriction S$. Denote the models N_α. $\mathcal{U}$ will actually be padded. Padding occurs just at every limit ordinal: $N_\alpha = N_{\sup(S \cap \alpha)}$ when $\alpha \in S$ is a limit. (Note: we allow $\mathcal{U}-\mathrm{pred}(\gamma+1) = \alpha$ but not $\mathcal{U}-\mathrm{pred}(\gamma+1) = \sup(S \cap \alpha)$.)

We'll define copy embeddings $\pi_\alpha : N_\alpha \to M_\alpha$. In case M_α is active, let $\psi_\alpha : \mathrm{Ult}_0(N_\alpha, F^{N_\alpha}) \to \mathrm{Ult}_0(M_\alpha, F^{M_\alpha})$ be the canonical map induced by π_α. Otherwise let $\psi_\alpha = \pi_\alpha$. (We'll have enough elementarity of π_α that this makes sense.) We have $\psi_\alpha \restriction \mathrm{OR}^{N_\alpha} = \pi_\alpha$. We'll maintan inductively on α that (φ_α): $\forall \gamma, \delta, \xi+1 \in (S \cap \alpha+1)$,

- Elementarity: π_γ is a near $\deg^{\mathcal{T}}(\gamma)$-embedding,
- Range: $\mathrm{rg}(\pi_\gamma) \supseteq (A)_\gamma$,
- $\mathcal{U}$'s extenders: $\psi_\xi(E^{\mathcal{U}}_\xi) = E^{\mathcal{T}}_\xi$ or else $E^{\mathcal{U}}_\xi = F^{N_\xi}$ and $E^{\mathcal{T}}_\xi = F^{M_\xi}$,
- Exact-ν-lh-preservation: $\psi_\xi(\nu_{E^{\mathcal{U}}_\xi}) = \nu_{E^{\mathcal{T}}_\xi}$ and $\psi_\xi(\mathrm{lh}(E^{\mathcal{U}}_\xi)) = \mathrm{lh}(E^{\mathcal{T}}_\xi)$,
- Half-ν-preservation: if $\xi < \gamma$ then $\pi_\gamma(\nu^{\mathcal{U}}_\xi) \geq \nu^{\mathcal{T}}_\xi$,

- Agreement: if $\xi < \gamma$ then ψ_ξ agrees with π_γ below $\nu^{\mathcal{U}}_\xi$; if $E^{\mathcal{U}}_\xi$ is type 1 or 2 or $\gamma = \xi + 1$ then they in fact agree below $\mathrm{lh}(E^{\mathcal{U}}_\xi) + 1$,

- Commutativity: if $\delta < \gamma$ and $i^{\mathcal{U}}_{\delta,\gamma}$ is defined then $\pi_\gamma \circ i^{\mathcal{U}}_{\delta,\gamma} = i^{\mathcal{T}}_{\delta,\gamma} \circ \pi_\delta$.

Set $N_0 = M_0 = M$ and $\pi_0 = \mathrm{id}$; clearly φ_0.

Suppose we have $\mathcal{U} \restriction S \cap \alpha + 1$, φ_α holds and $\alpha + 1 \in S$. First we define $E^{\mathcal{U}}_\alpha$ and then show that it is legal. If $E^{\mathcal{T}}_\alpha \in \mathrm{rg}(\pi_\alpha)$, set $E^{\mathcal{U}}_\alpha = \pi_\alpha^{-1}(E^{\mathcal{T}}_\alpha)$. If $E^{\mathcal{T}}_\alpha = F^{M_\alpha}$, set $E^{\mathcal{U}}_\alpha = F^{N_\alpha}$.

Otherwise since $(A)_\alpha \subseteq \mathrm{rg}(\pi_\alpha)$, 4.1 implies M_α is type 3, so let $\nu = \nu_{F^{M_\alpha}}$ and $\bar\nu = \nu_{F^{N_\alpha}}$. We must have $\nu < \mathrm{lh}(E^{\mathcal{T}}_\alpha) < \mathrm{OR}^{M_\alpha}$, so there are $a_{\mathrm{lh}}, f_{\mathrm{lh}} \in \mathrm{rg}(\pi_\alpha)$ with $\mathrm{lh}(E^{\mathcal{T}}_\alpha) = [a_{\mathrm{lh}}, f_{\mathrm{lh}}]^{M_\alpha}_{F^{M_\alpha}}$. This gives $E^{\mathcal{T}}_\alpha \in \mathrm{rg}(\psi_\alpha)$; set $E^{\mathcal{U}}_\alpha = \psi_\alpha^{-1}(E^{\mathcal{T}}_\alpha)$. Since "$[a, f]$ represents an ordinal not in my OR" is Π_1 and π_α is at least that elementary, $\psi_\alpha(\bar\nu) = \nu$ and

$$\psi_\alpha(\mathrm{OR}^{N_\alpha}) = \psi_\alpha((\bar\nu^+)^{\mathrm{Ult}(N_\alpha, F^{N_\alpha})}) = (\nu^+)^{\mathrm{Ult}(M_\alpha, F^{M_\alpha})} = \mathrm{OR}^{M_\alpha}. \tag{8}$$

So $E^{\mathcal{U}}_\alpha$ is on the N_α sequence.

We now show that $E^{\mathcal{U}}_\alpha$ is indexed above $\mathcal{U}$'s earlier extenders.

If $\alpha = \xi + 1$, by exact-ν-lh-preservation and agreement, $\pi_\alpha(\mathrm{lh}(E^{\mathcal{U}}_\xi)) = \mathrm{lh}(E^{\mathcal{T}}_\xi) < \mathrm{lh}(E^{\mathcal{T}}_\alpha)$, so $\mathrm{lh}(E^{\mathcal{U}}_\alpha)$ is certainly high enough.

Suppose α is a limit. Let $\beta' = \mathcal{T}\text{−pred}(\beta)$ where $\beta = \sup(S \cap \alpha)$ as in 4.1. Let $i^{\mathcal{T}}_{\beta',\alpha}(\mathrm{lh}') = \mathrm{lh}(E^{\mathcal{T}}_\alpha)$ (where $\mathrm{lh}' = \mathrm{OR}^{M_{\beta'}}$ is possible). Then as $\mathcal{T}$ is normal, $\mathrm{crit}(i^{\mathcal{T}}_{\beta',\alpha}) < \mathrm{lh}'$. So $\mathrm{lh}(E^{\mathcal{T}}_{\beta-1}) < i^{\mathcal{T}}_{\beta',\beta}(\mathrm{lh}')$. As in the successor case, we have $\pi_\beta(\mathrm{lh}(E^{\mathcal{U}}_{\beta-1})) = \mathrm{lh}(E^{\mathcal{T}}_{\beta-1})$, and since $\pi_\alpha = i^{\mathcal{T}}_{\beta,\alpha} \circ \pi_\beta$, the claim follows.

Unless N_α is active type 3 and $E^{\mathcal{U}}_\alpha = F^{N_\alpha}$, exact-$\nu$-lh-preservation for $\xi = \alpha$ is routine. But if so, there's a representation $[a_\nu, f_\nu]$ of $\nu = \nu^{M_\alpha}$ in $\mathrm{rg}(\pi_\alpha)$, and the argument leading to 8 works here too.

Letting $\gamma = \mathcal{T}\text{−pred}(\alpha + 1)$, we have $\gamma, \gamma + 1 \in S$. Let $\kappa = \mathrm{crit}(E^{\mathcal{U}}_\gamma)$. By half-$\nu$-preservation, $\kappa < \nu^{\mathcal{U}}_\gamma$. Exact-$\nu$-lh-preservation and the agreement between earlier embeddings and π_α imply that for $\gamma' < \gamma$ in S, $\nu^{\mathcal{U}}_{\gamma'} \le \kappa$, so setting $\gamma = \mathcal{U}\text{−pred}(\alpha + 1)$ is normal. Moreover, ψ_γ agrees with π_α below $(\kappa^+)^{N_\alpha}$. So the hypotheses for the shift lemma ([22], 4.2) apply (to the appropriate initial segments of N_γ, M_γ), which gives $\pi_{\alpha+1}$. As π_γ is a near $\deg^{\mathcal{U}}(\gamma)$-embedding, the shift lemma and ([15], 1.3) gives $\deg^{\mathcal{U}}(\alpha+1) = \deg^{\mathcal{T}}(\alpha+1)$ and that $\pi_{\alpha+1}$ is a near $\deg^{\mathcal{U}}(\alpha + 1)$-embedding. (Let $k = \deg^{\mathcal{T}}(\alpha + 1)$. Then by Σ_{k+1}-elementarity, $\pi_\gamma(\rho_k^{M^{\mathcal{T}}_\gamma}) \ge \rho_k^{M^{\mathcal{T}}_\gamma}$. Thus the $\mathcal{U}$ side doesn't drop in degree whilst the $\mathcal{T}$ side maintains.) It also gives the required commutativity, and that $\pi_{\alpha+1}$ agrees with ψ_α below $\mathrm{lh}(E^{\mathcal{U}}_\alpha)+1$, which yields half-$\nu$-preservation and agreement.

Now let $x \in A_{\alpha+1}$, and let a_x, q_x, τ_x be as in the definition of support. Inductively, we have $\bar a, \bar q$ so that $\pi_\gamma(\bar q) = q_x$ and $\pi_\alpha(\bar a) = a_x$. As $\deg^{\mathcal{U}}(\alpha + 1) = \deg^{\mathcal{T}}(\alpha + 1)$, $\bar x = [\bar a, f_{\tau_x, \bar q}]$ is an element of $N_{\alpha+1} = \mathrm{Ult}_{\deg^{\mathcal{U}}(\alpha+1)}((N)^*_{\alpha+1}, E^{\mathcal{U}}_\alpha)$. Moreover, by definition, $\pi_{\alpha+1}(\bar x) = x$. Therefore $\mathrm{rg}(\pi_{\alpha+1})$ is good. (Note: To maintain the range condition, it is essential here that the degrees of $\mathcal{U}$ are as large as those of $\mathcal{T}$. If we only had weak embeddings, the remark in the previous paragraph wouldn't apply, and it seems this might fail, though we don't have an example of such.) This gives $\mathcal{U} \restriction \alpha + 2$ and establishes $\varphi_{\alpha+1}$.

Now suppose we have $\mathcal{U}\restriction\beta+1$ for some $\beta\in S$, $\beta<\theta$, and $\varphi_{\beta+1}$ holds, but $\beta+1\notin S$. So $\inf(S-(\beta+1))$ is a limit α. Thus we set $E^{\mathcal{U}}_\beta=\emptyset$ and $N_\alpha=N_\beta$. Let $\pi_\alpha=i^{\mathcal{T}}_{\beta,\alpha}\circ\pi_\beta$. This makes sense and yields a near $\deg^{\mathcal{T}}(\alpha)=\deg^{\mathcal{U}}(\alpha)$-embedding since there is no dropping of any kind in $[\beta,\alpha]_T$. Its range is large enough since $\mathrm{rg}(\pi_\beta)\supseteq A_\beta$ and $i^{\mathcal{T}}_{\beta,\alpha}$"$A_\beta\supseteq A_\alpha$. By 4.1, β is a successor; let $E=E^{\mathcal{U}}_{\beta-1}$. By exact-$\nu$-lh-preservation,

$$\pi_\beta(\nu_E)=\nu^{\mathcal{T}}_{\beta-1}\leq\mathrm{crit}(i^{\mathcal{T}}_{\beta,\alpha}).$$

Therefore $i^{\mathcal{T}}_{\beta,\alpha}\circ\pi_\beta$ agrees with π_β below ν_E. Moreover, if E is type 1/2, then in fact

$$\pi_\beta(\mathrm{lh}(E))=\mathrm{lh}(E^{\mathcal{T}}_{\beta-1})<\mathrm{crit}(i^{\mathcal{T}}_{\beta,\alpha}),$$

so we get the stronger agreement. This easily extends to extenders used prior to E. Since $\pi_\alpha(\nu_E)\geq\pi_\beta(\nu_E)$, half-$\nu$-preservation is maintained. (But this is where we may lose exact-ν-lh-preservation.) This gives $\mathcal{U}\restriction\alpha+1$ and shows φ_α, completing the construction. □(Lemma 4.2)

Definition 4.3. Let $\mathcal{T}$, B be as in 4.2, and suppose A captures B. The finite support tree $\mathcal{T}^A_B$ for B, relative to A, is the tree $\mathcal{U}$ as defined in the proof of 4.2. Letting A^* be the support given by the algorithm described earlier, the finite support tree $\mathcal{T}_B$ is $\mathcal{T}^{A^*}_B$.

Total Measures

Definition 4.4. Given a pre-extender E over a premouse N, let $P_{N,E}$ (or just P_E where N is understood), denote the unique structure that might be a premouse with $\mathrm{tc}(E)$ active: $(\mathrm{Ult}(N,E)|(\nu_E^+)^{\mathrm{Ult}(N,E)},\mathrm{tc}(E))$.

Remark 4.5. Given E, note that P_E is a premouse iff it is wellfounded and it satisfies the initial segment condition.

Definition 4.6. Let E be an (possibly long) extender over κ. Then E is a *measure* or E is *finitely generated* if there is $s\in\nu_E^{<\omega}$ such that for all $\alpha<\nu_E$, there is $f:\kappa\to\kappa$, measured by E, such that $\alpha=[a,f]_E$.

E is a *total* measure if it is (equivalent to) an ω-complete non-principal ultrafilter.

Let $j:P\to R$ be an elementary embedding of premice with $\mathrm{crit}(j)$ inaccessible in P and $E_j\restriction j(\mathrm{crit}(j))\notin R$. Then the Dodd fragments of j are of *measure type* if $\sigma_j=(\mathrm{crit}(j)^+)^P$.

Lemma 4.7. *Let $j:P\to R$ be a fully elementary embedding of premice with $\kappa=\mathrm{crit}(j)$ inaccessible in P, and $P\models\mathsf{ZF}^-$. Suppose $E_j\restriction j(\kappa)\notin R$. Let $\mathcal{U}$ be a normal iteration tree on R, such that $i^{\mathcal{U}}$ exists and $\mathrm{crit}(i^{\mathcal{U}})>\kappa$. Then $i^{\mathcal{U}}(s_j)$, $\sup i^{\mathcal{U}}$"σ_j are the Dodd-fragment ordinals of $i^{\mathcal{U}}\circ j$.*

Proof. Consider a single ultrapower of R by an extender F which is close to R (but maybe not in R), with $\mu=\mathrm{crit}(F)>\mathrm{crit}(j)$. Since $R\models\mathsf{ZF}^-$, $\mathrm{Ult}_\omega(R,F)=\mathrm{Ult}_0(R,F)$. So if the ultrapower has too large a fragment of $E_{i_F\circ j}$, there's a function $f:\mu\to R$ in R and $a\in\nu_F^{<\omega}$

such that $[a, f]$ represents it. Also since $R \models \mathsf{ZF}^-$, the closeness of F to R implies $F_a \in R$. For $\alpha < \sigma_j$, there's F_a-measure one many u such that

$$f(u) \restriction \alpha \cup s_j = E_j \restriction \alpha \cup s_j.$$

It follows that $E_j \restriction \sigma_j \cup s_j \in R$; contradiction. This extends to normal iterates of R. $\square$(Lemma 4.7)

Theorem 4.8. *Let N be an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse, $E, \kappa \in N$, and*

$$N \models \mathsf{KP}^*, \ E \text{ is a total wellfounded measure on } \kappa \text{ and } \kappa^{++} \text{ exists.}$$

Then there is a iteration tree $\mathcal{T}$ on N with the following properties:

(a) $\mathcal{T}$ is a finite, normal tree based on $N|(\kappa^{++})^N$.

(b) $\mathrm{Ult}(N, E)$ is the last model of $\mathcal{T}$ (so $\mathcal{T}$ results from comparing N with $\mathrm{Ult}(N, E)$), and there is no dropping on $\mathcal{T}$'s main branch.

(c) E is the extender of $i^{\mathcal{T}}$.

(d) There is a finite linear iteration $\mathcal{L}$ of N, possibly non-normal, with final model $\mathrm{Ult}(N, E)$ and $i^{\mathcal{L}} = i^{\mathcal{T}}$.

(e) $\mathfrak{C}_D(E)$ is a measure and is the active extender of $P \triangleleft N$.

(f) For any F used in $\mathcal{T}$, F and $\mathfrak{C}_D(F)$ are measures.

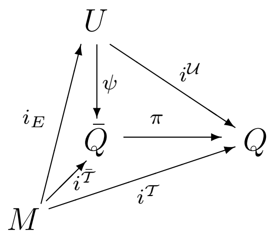
(g) P_E satisfies the initial segment condition iff there is exactly one extender used along $\mathcal{T}$'s main branch. (Clearly that extender must be E.)

(h) The extenders used in $\mathcal{T}$ are characterized in Corollary 4.10.

Proof. The proof is by induction on mice, so we assume that proper levels of $N|(\kappa^{++})^N$ satisfy the theorem. Conclusions (d) and (h) won't be needed in the induction, however. Let $M = \mathrm{Hull}_\omega(N|(\kappa^{++})^N)$. The failure of the theorem is first order over $N|(\kappa^{++})^N$, so as usual we assume E is the least. Let $\bar{E}$ be the collapse of E. A simplification of the argument at the start of the proof of 2.9 shows $\mathrm{Ult}(M, \bar{E})$ embeds into a level of N, so is iterable.

Notation. Again we have no further need for the N-level, so we'll drop the bar notation.

As in 2.9, comparing M with $\mathrm{Ult}(M, E)$ results in trees $\mathcal{T}$ on M and $\mathcal{U}$ on $\mathrm{Ult}(M, E)$ with a common last model Q. Let s generate E. Let $\bar{\mathcal{T}} = \mathcal{T}_{i_{U,Q}(s)}$ be the finite support tree for $i_{U,Q}(s)$ relative to $\mathcal{T}$, as in 4.3. Let $\bar{Q}$ be $\bar{\mathcal{T}}$'s final model, and $\tau : \bar{Q} \to Q$ be the liftup map given by 4.2. Since $i^{\mathcal{U}}(s) \in \mathrm{rg}(\pi)$, we have the natural factor map $\psi : U \to \bar{Q}$. The following diagram commutes:



Claim 1. $\mathrm{crit}(i^{\mathcal{T}}) = \mathrm{crit}(i^{\bar{\mathcal{T}}}) = \mathrm{crit}(E) < \mathrm{crit}(i^{\mathcal{U}})$. *Therefore* $\mathrm{crit}(E) < \mathrm{crit}(\psi)$ *and* $\mathrm{crit}(E) < \mathrm{crit}(\pi)$.

Proof. Since the diagram commutes, otherwise $\kappa = \mathrm{crit}(i^{\mathcal{T}}) = \mathrm{crit}(i^{\mathcal{U}}) \leq \mathrm{crit}(E)$, and $i^{\mathcal{T}}$ must be compatible with $i^{\mathcal{U}}$ on $\mathcal{P}(\kappa)$ through $\inf\{i^{\mathcal{T}}(\kappa), i^{\mathcal{U}}(\kappa)\}$, since M is its own hull. But $(\mathcal{T},\mathcal{U})$ was a comparison, so this is false. The second statement now follows commutativity. □(Claim 1)

Claim 2. *We may assume that for every extender F used in $\bar{\mathcal{T}}$, F and $\mathfrak{C}_D(F)$ are measures. Moreover, letting E^* be the (long) extender of $\bar{\mathcal{T}}$'s main branch embedding, we may assume that $\bar{\mathcal{T}}$ witnesses all conclusions of the theorem other than (h), with respect to E^*.*

We'll first assume the claim, and finish the proof. Let G be the first extender used on $\bar{\mathcal{T}}$'s main branch. By Claim 2, $\mathfrak{C}_D(G)$ is a measure, so by 2.17, the Dodd fragments of G, and so those of $i^{\bar{\mathcal{T}}}$ (in $\bar{Q}$), are of measure type. Consider the Dodd fragments of E and $i^{\mathcal{T}}$. By commutativity and 1, ψ maps fragments of E to fragments of $i^{\bar{\mathcal{T}}}$, and likewise π from $i^{\bar{\mathcal{T}}}$ to $i^{\mathcal{T}}$ and $i^{\mathcal{U}}$ from E to $i^{\mathcal{T}}$. By 4.7, since $i^{\mathcal{U}}$ is an iteration, it also preserves the *maximality* of Dodd fragments. By commutativity, ψ and π preserve maximality also. In particular, the fragments of i_E and $i^{\mathcal{T}}$ are also of measure type.

Let $\langle F_\alpha \rangle$ enumerate the Dodd cores of the extenders $F \leq_{\mathrm{dam}} G$ for the extenders G used on $\mathcal{T}$'s main branch, in order of increasing critical points. Let $\langle Q_\alpha, j_{\alpha,Q} \rangle$ be the corresponding core sequence (if $\mathcal{T}$'s main branch uses more than one extender, simply string the corresponding core sequences together). Applying 3.20 to each of the extenders used on $\mathcal{T}$'s main branch, we know the core sequence reaches all the way to Q, and the analogue to the characterization following 3.20 also holds. Let $\langle \bar{F}_\alpha \rangle$ and $\langle \bar{Q}_\alpha \rangle$ enumerate the corresponding objects for $\bar{\mathcal{T}}$. Claim 2 gives $\bar{F}_\alpha$ is a measure. From the previous paragraph, we have $\bar{F}_0 = F_0$, and $i_E, i^{\bar{\mathcal{T}}}$ and $i^{\mathcal{T}}$ factor through $\bar{Q}_1 = Q_1 = \mathrm{Ult}(M, F_0)$, commutatively. (It doesn't matter whether κ is an $s_{i^{\mathcal{T}}}$-generator, since $\kappa < \mathrm{crit}(i^{\mathcal{U}})$ anyway.)

Suppose $Q_1 \neq Q$. As in 3.21, $j_1 : Q_1 \to Q$ is compatible with an extender used on the M side of the comparison, through its sup of generators. As in Claim 1, it follows that $\mathrm{crit}(j_1) < \mathrm{crit}(i^{\mathcal{U}})$, and the rest of the above argument can be repeated. Maintaining this situation inductively, we eventually (in finitely many stages) reach $U = \bar{Q}$, so U is a finite iterate of M, so $U = Q = \bar{Q}$, and $\mathcal{T} = \bar{\mathcal{T}}$ is finite and as desired. So to finish proving the theorem we just need:

Claim 2 Proof.

Notation. For this proof we need only refer to the finite tree, which we'll refer to as $\mathcal{T}$ instead of $\bar{\mathcal{T}}$ from now on.

The motivation for the proof is to refine the finite support tree construction. Suppose $\mathcal{T}$ uses an extender E which isn't a measure. Since we only need to support finitely much, the intuition is that only finitely many of E's generators are important, so we should be able to improve $\mathcal{T}$ by replacing E with a sub-measure. Doing this to every such extender should yield the type of tree we want. However, executing this takes some care. One problem is that it seems interfering with some part of the tree in this way might affect the normality later on. To get around this, we start from the end of $\mathcal{T}$ and work backwards, producing a series of trees $\langle \mathcal{T}_i \rangle$, converting extenders to measures one by one. $\mathcal{T}_{i+1}$ will replicate $\mathcal{T}$ until (a version of) the relevant E first appears, then convert the use of E to the production and use of a sub-measure, ensuring that the sub-measure includes enough generators to allow a "downward" copying of the remainder of $\mathcal{T}_i$. Because we've already processed that remainder, it only involves simple interactions with the earlier part of the tree.

So, $\mathcal{T}$ is finite. Let $F_1, \ldots, F_n$ enumerate the extenders $F \leq_{\text{dam}} G$ for any G used along the main branch of $\mathcal{T}$, this time in order of *de*creasing critical point. Note that F_1 is Dodd sound. If it's a measure, we set $\mathcal{T}_1 = \mathcal{T}$. Suppose otherwise. Say F_1 is on the P sequence and $\kappa = \text{crit}(F_1)$. After using $F_1 = G_1$, $\mathcal{T}$ just pops out of the damage structure, hitting a sequence of active extenders $G_2, \ldots, G_k$, where G_i is largest in $\text{dam}(G_{i+1})$, until producing its final model $Q = \text{Ult}_\omega(M_p^{\mathcal{T}}, G_k)$ ($k = 1$ is possible). Let $a_k \in \nu_{G_k}^{<\omega}$ be sufficient to support x in this ultrapower. Let a_{i-1} be sufficient to support a_i, and $a \in \tau_{F_1}^{<\omega}$ be such that $a \cup t_{F_1}$ is sufficient to support a_2 and $F_1 \restriction (\max(t_{F_1}))$. (If F_1 is type 3 then instead let $\max(a)$ be a generator which indexes a segment of F_1; these are unbounded.) Since $(\kappa^+)^{F_1} < \tau_{F_1}$ (and τ_{F_1} is a cardinal of $P|lh(F_1)$), Dodd soundness implies $\mu = F_1 \restriction (a \cup t_{F_1}) \in P|\tau_{F_1}$. Considering the natural factor map $\text{Ult}_0(P|\text{lh}(F_1), \mu) \to \text{Ult}_0(P|\text{lh}(F_1), F_1)$, it's easy to see that our choice of a implies μ has the initial segment condition. By the minimality of M, there's a finite tree $\mathcal{U} = \mathcal{U}_1$ on a proper segment of $P|(\kappa^{++})^{F_1}$, which uses only measures, such that μ is on the sequence of its final model. (Use condensation to get a proper level of $P|(\kappa^{++})^{F_1}$ containing μ_1 to apply 4.8 to.)

Now let q be least such that $\text{lh}(E_q^{\mathcal{T}}) > (\kappa^+)^{F_1}$. We would like to have $\mathcal{T}_1$ begin by followwing $\mathcal{T}$ until reaching $M_q^{\mathcal{T}}$, then followwing $\mathcal{U}$. This will be fine unless $\nu_{q-1}^{\mathcal{T}} > \kappa$ and $\mathcal{U}$ uses an extender with crit κ; in this case $\mathcal{T} \restriction q + 1 {}^\frown \mathcal{U}$ isn't normal. To deal with this we need to observe some properties of $\mathcal{U}$. First, $\mathfrak{C}_D(\mu_1)$ is on the P-sequence by 4.8. We first claim that $\mathcal{U}$ is equivalent to a tree on $P|\text{lh}(\mathfrak{C}_D(\mu_1))$.

If $E_0^{\mathcal{U}}$ exists then $(\kappa^+)^\mu < \text{lh}(E_0^{\mathcal{U}})$. If μ isn't the active extender of $\mathcal{U}$'s last model then, since μ projects to $(\kappa^+)^\mu$, in fact μ is on the P-sequence. Assume it is active. As $\text{crit}(\mu) = \kappa$, $\text{crit}(\mathfrak{C}_\omega(\mu) \to \mu > \kappa$. As $\mathfrak{C}_D(\mu)$ is a measure, so is $\mathfrak{C}_\omega(\mu)$, so they're equal (by the Dodd-fragment preservation facts), and on the P-sequence. Also $\text{lh}(E_0^{\mathcal{U}}) \leq \text{lh}(\mathfrak{C}_D(\mu))$. So with $i^{\mathcal{U}} : \mathfrak{C}_D(\mu) \to \mu$ the (dropping) branch map, $\text{crit}(i^{\mathcal{U}}) > \kappa$. Because $\mathcal{U}$ uses only measures,

$M_0^{\mathcal{U}}$ is passive, and κ is a cardinal of P, we get all of $\mathcal{U}$'s crits are $\geq \kappa$.

Now suppose $\mathrm{crit}(E_i^{\mathcal{U}}) = \kappa$. Then $(M^*)_{i+1}^{\mathcal{U}} = P$. Since $\mathrm{crit}(i^{\mathcal{U}}) > \kappa$, $M_{i+1}^{\mathcal{U}}$ isn't on the main branch, so $\mathcal{U}$ goes back at some point. Because it only uses measures, in fact it goes back to $M_i^{\mathcal{U}}$ at say stage j with $\mathrm{crit}(E_j^{\mathcal{U}}) = |\nu_i^{\mathcal{U}}|^{E_i^{\mathcal{U}}}$, and $\mathrm{lh}(E_{i+1}^{\mathcal{U}}) < (|\nu_i^{\mathcal{U}}|^{++})^{M_{i+1}^{\mathcal{U}}}$. So $\mathcal{U} \restriction [i+1, j]$ is on $M_{i+1}^{\mathcal{U}}|\mathrm{lh}(E_{i+1}^{\mathcal{U}})$ and $E_j^{\mathcal{U}}$ triggers a drop to $M_i^{\mathcal{U}}|\mathrm{lh}(E_i^{\mathcal{U}})$. It follows that $\mathcal{U}$ can be considered a tree on $P|\mathrm{lh}(\mathfrak{C}_D(\mu))$, or else on $M_q^{\mathcal{T}}$. (One can also show that if $\mathrm{crit}(E_i^{\mathcal{U}}) = \kappa$, then $E_i^{\mathcal{U}}$ is the image of $\mathfrak{C}_D(\mu)$.)

Now consider $\mathcal{T} \restriction (q+1) ^\frown \mathcal{U}$, and suppose $\kappa < \nu_{q-1}^{\mathcal{T}}$ and $\mathrm{crit}(E_i^{\mathcal{U}}) = \kappa$. $\mathcal{U}$ applies this extender to $M_0^{\mathcal{U}} = M_q^{\mathcal{T}}$, but for normality, the correct model to return to is $M_{q-1}^{\mathcal{T}}$, with $(M^*)_{i+1} = M_{q-1}^{\mathcal{T}}|(\kappa^+)^{F_1}$, and since $\rho_1^{(M^*)_{i+1}} = \kappa$ the correct degree is 0. The preceding paragraph shows that this is fine; i.e. there is a normal tree $\mathcal{T} \restriction q+1 ^\frown \mathcal{U}'$ such that $\mathcal{U}'$ is given by copying the extenders of $\mathcal{U}$, and returning to $M_{q-1}^{\mathcal{T}}$ when κ is the crit. This tree also has μ on the sequence of its final model.

So let $\mathcal{T}_1 = \mathcal{T} \restriction q+1 ^\frown \mathcal{U}_1' ^\frown \mathcal{V}_1$, where $\mathcal{V}_1$ is a "downward copy" of the remainder of $\mathcal{T}$, hitting μ, then active extenders $G_2', \ldots, G_k'$, yielding a final model Q'. (Note μ and F_1 apply (normally) to the same model, which yields premice with active extenders G_2' and G_2 respectively. Moreover, G_2' is a sub-extender of G_2, they both apply to the same model, etc.) Since G_k' is a sub-extender of G_k, we get a $\pi_1 : Q_1 \to Q$ which is fully elementary, commuting with the $\mathcal{T}_1$ and $\mathcal{T}$ embeddings, and $x \in \mathrm{rg}(\pi_1)$ by our choice of generators a.

The general case is a little more complicated; we just sketch it. Suppose we have $\mathcal{T}_i$ where $1 \leq i < n$; first we describe some of our inductive assumptions. Let q be least such that $\mathrm{lh}(E_q^{\mathcal{T}}) > (\kappa_i^+)^{F_i}$. Then $\mathcal{T}_i$ is of the form $\mathcal{T} \restriction q+1 ^\frown \mathcal{U}_i$. If $E_i <_{\mathrm{dam}} E_{i+1}$ then we'll have that $\mathcal{U}_i$ uses a sub-extender E_{i+1}^i of E_{i+1}, whose damage structure consists only of measures, and that $\mathfrak{C}_D(E_{i+1}^i) = F_{i+1}$. Otherwise $\mathcal{T} \restriction q+1$ uses $E_{i+1}^i = E_{i+1} = F_{i+1}$. Moreover, $\mathcal{U}_i$'s extenders with crit $\geq \tau_{F_{i+1}}$ are all measures, and $\mathcal{U}_i$'s crits below $\tau_{F_{i+1}}$ are all of the form κ_m for some m.

Now we construct $\mathcal{T}_{i+1}$. Note that F_{i+1} is on the sequence of $M_{q'}^{\mathcal{T}}$ for some $q' \leq q$. (Either $E_i <_{\mathrm{dam}} E_{i+1}$ so the critical point κ_i applies to $M_{q'}^{\mathcal{T}}$, or $E_{i+1} = F_{i+1}$ is Dodd-sound and $\nu_{E_{i+1}} \leq \kappa_i$.) If F_{i+1} is itself a measure, let $\mathcal{T}_{i+1} = \mathcal{T}_i$. Otherwise, as in the $i = 0$ case, we choose $a \in \tau_{F_{i+1}}$ such that $t_{F_{i+1}} \cup a$ is sufficient to (eventually) generate the remainder of $\mathcal{T}_i$, the $\mathcal{T}_i$ preimage of $x \in Q$, the Dodd-solidity witnesses for F_{i+1} and $\tau_{F_{i+1}}$. (If F_{i+1} is type 3, choose $\max(a)$ as in the $i = 0$ case and work relative to $F_{i+1} \restriction \max(a)+1$.) Let $\mu_{i+1} = F_{i+1} \restriction a \cup t_{F_{i+1}}$. As before, μ_{i+1} is on the sequence of the last model of a tree $\mathcal{U}_{i+1}^0$ on $P|(\kappa_{i+1}^{++})^{F_{i+1}}$. Let q_{i+1} be least such that $\mathrm{lh}(E_{q_{i+1}+1}^{\mathcal{T}}) > (\kappa_{i+1}^+)^{F_{i+1}}$. Then $\mathcal{T}_{i+1} = \mathcal{T} \restriction (q_{i+1}+1) ^\frown (\mathcal{U}_{i+1}^0 {}^\frown \mathcal{U}_{i+1}^1)' {}^\frown \mathcal{U}_{i+1}^2$; we'll now define $\mathcal{U}_{i+1}^1$ and $\mathcal{U}_{i+1}^2$. (Here $\mathcal{V}'$ is the modification of $\mathcal{V}$ we used in the $i = 0$ case.)

If $E_i \not<_{\mathrm{dam}} E_{i+1}$ then E_{i+1} is the "top" extender in the damage structure of the extender used on a branch immediately preceding E_i. I.e. let q' be such that E_i applies to $M_{q'}^{\mathcal{T}}$. Suppose $E_i \in \mathrm{dom}(\mathrm{dam}(E_m))$. Then $E_{i+1} <_{\mathrm{dam}} E_m$. Immediately after using E_{i+1}, $\mathcal{T}$ successively applies E_k for all k with $E_{i+1} <_{\mathrm{dam}} E_k <_{\mathrm{dam}} E_m$ (in their nested order), which results in $M_{q'}^{\mathcal{T}}$. Clearly either $q' = q_i$ or $\mathrm{lh}(E_{q'}^{\mathcal{T}}) = (\kappa_i^+)^{F_i}$. In the latter case, E_i triggers a drop to $M^{\mathcal{T}}_{q'}|\mathrm{lh}(E^{\mathcal{T}}_{q'})$, so $E^{\mathcal{T}}_{q'}$ is on the branch leading from $\mathfrak{C}_D(E_m)$ to E_m, so $\mathrm{crit}(E^{\mathcal{T}}_{q'}) = \kappa_m$. In fact $E_{i+1} <_{\mathrm{dam}} E^{\mathcal{T}}_{q'}$, so $E^{\mathcal{T}}_{q'}$ is the active extender of $M^{\mathcal{T}}_{q'}$. Alternatively E_i may be applied directly along $\mathcal{T}$'s main branch. In this case E_i can't trigger a drop, so $q' = q_i$. So set $\mathcal{U}^1_{i+1} = \emptyset$, and as in the $i = 0$ case, set $\mathcal{U}^2_{i+1}$ to copy down the above activity by hitting μ_{i+1}, then the resulting active extenders, until reaching a preimage of $M^{\mathcal{T}}_{q_i}$, and then to copy down $\mathcal{U}_i$. Notice that we've maintained the inductive restrictions on $\mathcal{T}_{i+1}$'s extenders and crits.

Otherwise E_i is the least extender in $\mathrm{dam}(E_{i+1})$. Using the natural map reducing $\mu = \mu_{i+1}$ to F_{i+1}, let $\mathcal{U}^1_{i+1}$ be the downward copy of the segment of $\mathcal{T}_i$ damaging F_{i+1} (which is essentially on $M^{\mathcal{T}_i}_p|\mathrm{lh}(F_{i+1})$ where $p = q_i$ or $p = q_i - 1$), to the last model of $\mathcal{U}^0_{i+1}$ (so $\mathcal{U}^1_{i+1}$ is on that last model). We will show $\mathcal{U}^0_{i+1} {}^\frown \mathcal{U}^1_{i+1}$ is normal. Let $\pi : \mu \to F_{i+1}$ be the reduction map and $\pi(\bar{\tau}) = \tau_{F_{i+1}}$. We claim that for all extenders F used in $\mathcal{U}^0_{i+1}$, $\nu_F \leq \bar{\tau}$. Let $t = t_{F_{i+1}} = s_{F_{i+1}}$ and $\pi(\bar{t}) = t$. Note $\bar{t} \subseteq s_\mu$ by our choice of a. Let $j = i^{\mathcal{U}^0_{i+1}}$. By 4.8, as μ has the initial segment condition, $\mu_{\mathfrak{C}} = \mathfrak{C}_D(\mu)$ is on the P-sequence, is the domain of j, and by the Dodd-fragment preservation facts, $j(s_{\mu_{\mathfrak{C}}}) = s_\mu \supseteq \bar{t}$. Since $\bar{t} \cup \bar{\tau}$ generates μ, it follows that all generators along $\mathcal{U}^0_{i+1}$'s main branch are below $\bar{\tau}$. As $\mathcal{U}^1_{i+1}$ only uses measures, its crits are at least τ or exactly κ. (note $\tau \leq \kappa_i < \nu_F$ where F is the first extender used). The normality of $\mathcal{U}^0_{i+1} {}^\frown \mathcal{U}^1_{i+1}$ follows. Then $\mathcal{U}^2_{i+1}$ copies down the remainder of $\mathcal{U}_i$. Again we've maintained the inductive restrictions on extenders.

Finally, having gotten this far, the reader should be happy to check that our measure-only tree $\mathcal{T}_{\mathrm{meas}}$ also satisfies the other conclusions of the theorem with respect to its (long) main branch extender (which is a measure itself). The linear tree $\mathcal{L}$ of (d) is given by applying all extenders in the damage structures of the measures used along the main branch of $\mathcal{T}_{\mathrm{meas}}$, but in order of increasing critical point, not length.

□(Claim 2)(Theorem 4.8)

Remark 4.9. During the proof, we dealt with the possibility that $\mathcal{T}$ uses extenders not in the damage structure of E. This does occur; for example, suppose $N \models \mathsf{ZFC}$ is a mouse, and F is a finitely generated total extender on $\mathbb{E}^N$. Let κ be the largest cardinal of $N|\mathrm{lh}(F)$. Suppose κ is measurable in $\mathrm{Ult}(N, F)$, and D is a witnessing normal measure. Then $E = D \circ F$ is a measure of N, and the resulting tree $\mathcal{T}$ uses 3 extenders, $E^{\mathcal{T}}_0 = F$, $E^{\mathcal{T}}_1 = D$, and $E^{\mathcal{T}}_2 = E$. But the damage structure of E just involves E and D. This is (a simple case of) the only exception - we leave the proof of the following corollary to the reader.

Corollary 4.10. *With $\mathcal{T}$ as in 4.8, let $F_1, \ldots, F_r$ be the extenders used along the main branch of $\mathcal{T}$. Suppose $E^{\mathcal{T}}_m$ exists and let $P = M^{\mathcal{T}}_m|\mathrm{lh}(E^{\mathcal{T}}_m)$. Then $\mathfrak{C}_D(E^{\mathcal{T}}_m)$ is the active extender of $\mathfrak{C}_\omega(P)$. Also there are unique n, k such that $\mathfrak{C}_D(E^{\mathcal{T}}_m) = \mathfrak{C}_D(E^{\mathcal{T}}_n)$, $m \leq_{\mathcal{T}} n$, and $E^{\mathcal{T}}_n \leq_{\mathrm{dam}} F_k$.* □

Submeasures

We now move on to consider submeasures of normal measures in mice, and prove some condensation-like facts in this context. Suppose N is a type 1 mouse and $\kappa = \mathrm{crit}(F^N)$.

Given $\mathcal{A} \subseteq \mathcal{P}(\kappa)^N$ of size κ in N, we show that the submeasure $F^N \restriction \mathcal{A}$ is often on $\mathbb{E}^N$. The basic structure of the proofs are like the main proofs in the earlier sections, in that we'll compare N with a phalanx derived from it and the submeasure. It was Steel's idea to use this approach here. Establishing the iterability of the phalanx is simple for 4.11 and 4.12, as it is inherited directly from the mouse's active normal measure. In the case of 4.15 there are fine structural complications.

Theorem 4.11. *Let M be an $(0, \omega_1 + 1)$-iterable type 1 mouse, with active measure μ, with crit κ. Let $\kappa < \beta < (\kappa^+)^M$ with $M|\beta$ passive, and $\bar{\mu} = \mu \cap M|\beta$. Suppose* $\mathrm{Ult}_0(M|\beta, \bar{\mu}) \models \beta = \kappa^+$. *Then $\bar{\mu}$ is on the M sequence.*

Proof. Since the failure of the theorem is a Σ_1 fact about M, we may assume $\rho_1^M = \omega$ and M is 1-sound (replacing M with $\mathrm{Hull}_1^M(\emptyset)$ if necessary).

Let γ be least such that $\beta \leq \gamma$ and $M|\gamma$ projects to κ. (So in fact $\beta < \gamma$ since $M|\beta \models \mathsf{ZF}^-$.) Let k be largest such that $\rho_{k+1}^{M|\gamma} \leq \kappa < \rho_k^{M|\gamma}$. Note that $\mathrm{Ult}_k(M|\gamma, \bar{\mu})$ makes sense (i.e. $\bar{\mu}$ measures its subsets of κ, and if $M|\gamma$ is active with a type 3 extender E, then $\kappa < \beta \leq \nu_E$, since $\beta = \kappa^+$ in $M|\gamma$), and agrees with $\mathrm{Ult}_0(M|\beta, \bar{\mu})$ beyond β, so is passive at β.

Claim 1. $\mathrm{Ult}_k(M|\gamma, \bar{\mu})|\beta = M|\beta$ *and the phalanx* $\mathcal{P} = (M, \mathrm{Ult}_k(M|\gamma, \bar{\mu}), \beta)$ *is* ω_1+1*-iterable.*

Proof. An iteration on this phalanx can be reduced to a freely dropping iteration on M, by reducing to a freely dropping iteration on the phalanx $(M, i_\mu(M|\gamma), (\kappa^+)^M)$. Let $\pi : \mathrm{Ult}_k(M|\gamma, \bar{\mu}) \to i_\mu(M|\gamma)$ be the natural factor map. Then π is a weak k-embedding, $\pi \circ i_{\bar{\mu}} = i_\mu \restriction M|\gamma$, $\pi \restriction \beta = \mathrm{id}$, and $\pi(\beta) = (\kappa^+)^M$. So $\mathrm{Ult}_k(M|\gamma, \bar{\mu})|\beta = M|\beta$. Moreover, we may use $\mathrm{id} : M \to M$ and π as initial copy maps to copy an iteration up. If E_α has crit κ, then so does $\pi_\alpha(E_\alpha)$. In this case E_α measures exactly $\mathcal{P}(\kappa) \cap M|\gamma$, while $\pi_\alpha(E_\alpha)$ measures $\mathcal{P}(\kappa) \cap M$. So there is a drop in model below, to $M|\gamma$, but no drop above, and $\pi_{\alpha+1} : M_{\alpha+1} \to i_{\pi_\alpha(E_\alpha)}(M|\gamma)$. Upon leaving $N_{\alpha+1}$, we impose the appropriate drop in model and degree. □(Claim 1)

By the first part of the claim, a comparison between M and $\mathcal{P}$ begins above β, so by the second part, there is a successful comparison, giving trees $\mathcal{T}$ and $\mathcal{U}$ respectively. Say $\mathcal{T}$ has last model N and $\mathcal{U}$ has last model Q. Q isn't fully sound. (The Closeness Lemma ([11], 6.1.5) doesn't show extenders applied to $M|\gamma$ (or all of M) are close, so the usual fine structure preservation arguments don't show ρ_{k+1} is preserved when an extender hits $M|\gamma$. However enough preservation holds that Q isn't sound.) It can't be that $N \triangleleft Q$, since $M = \mathrm{Th}_1^M(\emptyset)$. So $Q = N$. By almost usual arguments, Q is above U in $\mathcal{U}$. (Add the observation that if Q is above $M|\gamma$, in that the first extender used on the main branch of $\mathcal{U}$ has crit κ, then in fact $\rho_{k+1}^Q = \rho_{k+1}^{M|\gamma} = \kappa$, since Q also results from $\mathcal{T}$, which starts above β, and drops in model on its main branch.) As Q isn't sound, $b^\mathcal{T}$ drops, so $b^\mathcal{U}$ doesn't. So $\mathfrak{C}_{k+1}(Q) = M|\gamma$. So $b^\mathcal{T}$'s last drop is to $M|\gamma$ and $\mathrm{lh}(E_0^\mathcal{T}) \leq \gamma$. Let $E_\alpha^\mathcal{T}$ be the extender hitting $M|\gamma$ along $b^\mathcal{T}$. Then $E_\alpha^\mathcal{T}$ is compatible with the core embedding, and since the exchange ordinal of $\mathcal{P}$ is $\beta > \kappa$, this is compatible with $\bar{\mu}$ (and they measure the same sets). Therefore $\bar{\mu}$ is the normal measure derived from $E_\alpha^\mathcal{T}$. Finally, all extenders used had length above β, so $\bar{\mu}$ is in fact on the M sequence. □(Theorem 4.11)

Theorem 4.12. *Let M be an $\omega_1 + 1$-iterable type 1 mouse, with active measure μ, with critical point κ. Let $\kappa < \beta < (\kappa^+)^M$ be such that $M|\beta$ has largest cardinal κ, and is active with a type 2 extender, and let $\bar{\mu} = \mu \cap M|\beta$. Let $U = \mathrm{Ult}_0(M|\beta, \bar{\mu})$. Suppose $U|\beta \neq M|\beta$. Then $\bar{\mu}$ is on the $\mathrm{Ult}_0(M, E^M_\beta)$ sequence.*

Remark 4.13. It is the case that $U||\beta = M||\beta$, as μ coheres with M, and since $\beta = \kappa^+$ in $\mathrm{Ult}_0(M, E^M_\beta)$, there's enough closure below β that $\mathrm{Ult}_0(M|\beta, \bar{\mu})$ agrees with $\mathrm{Ult}_0(M, \mu)$ below β. So the hypothesis $U|\beta \neq M|\beta$ says that either $U|\beta$ is passive, or $E^U_\beta \neq E^M_\beta$. The conclusion of the theorem shows that in fact $U|\beta$ is passive.

Remark 4.14. The hypothesis $U|\beta \neq M|\beta$ doesn't follow from the other hypotheses, so is necessary. For suppose β is least satisfying the other hypotheses. Then $\beta = [\{\kappa\}, f]^M_\mu$ where f is definable over $M|\kappa$. Moreover, since $M|\beta$ projects to κ, the same holds for any $\alpha < \beta$. But then $\mathrm{Ult}_0(M|\beta, \bar{\mu})|\beta = M|\beta$.

Proof of Theorem 4.12. This is just as in the previous case, except that now $\beta = \gamma$, and $E^{\mathcal{T}}_0 = E^M_\beta$ (where $\mathcal{T}$ is the tree on M). This means that all extenders used on either side of the comparison with critical point κ measure exactly $M|\beta$. Since E^M_β is type 2, there's no problem taking an ultrapower of $M|\beta$ with an extender whose critical point is κ. □(Theorem 4.12)

Theorem 4.15. *Let M be a type 1 mouse with active measure μ, and suppose that $\mathrm{Ult}_0(M, \mu)$ is $(0, \omega_1 + 1)$-iterable. Let $\kappa = \mathrm{crit}(\mu)$. Let $\kappa < \beta < (\kappa^+)^M$ be such that $M|\beta$ has largest cardinal κ, is active with a type 3 extender, and let $\bar{\mu} = \mu \cap M|\beta$. Suppose $\bar{\mu}$ coheres with the $\mathrm{Ult}(M, E^M_\beta)$ sequence (meaning if $U = \mathrm{Ult}(M||\beta, \bar{\mu})$, and $\lambda = (\kappa^{++})^U$, then $U|\lambda = \mathrm{Ult}(M, E^M_\beta)||\lambda$, so in particular, $\beta = (\kappa^+)^U$). Then $\bar{\mu}$ is on the $\mathrm{Ult}(M, E^M_\beta)$ sequence.*

Proof. Again we assume $M = \mathrm{Hull}^M_1(\emptyset)$. Let $P = \mathrm{Ult}(M, E^M_\beta)$ and

$$\mathcal{P} = (M, \mathrm{Ult}(P, \bar{\mu}), \kappa + 1).$$

Note $\lambda = \kappa^{++}$ in $\mathrm{Ult}(P, \bar{\mu})$.

One must be a little careful with iterations of $\mathcal{P}$. Say $\mathcal{U}$ is on $\mathcal{P}$, $\beta < \mathrm{lh}(E^{\mathcal{U}}_0)$ and $\mathrm{crit}(E^{\mathcal{U}}_\gamma) = \kappa$. Then $E = E^{\mathcal{U}}_\gamma$ goes back to M, but measures only $\mathcal{P}(\kappa) \cap M|\beta$, so applies exactly to $M|\beta$. But $M|\beta$ is type 3, and κ is the height of $(M|\beta)^{\mathrm{sq}}$, so we can't form $\mathrm{Ult}((M|\beta)^{\mathrm{sq}}, E)$. This problem arose in the proof of condensation, ([22], 5.1). We need to generalize what's done there: just set $M^{\mathcal{U}}_{\gamma+1} = \mathrm{Ult}_0(M|\beta, E)$, where the ultrapower is formed without squashing, then carry on as usual. $F^{M|\beta}$ is shifted to F^{Ult} in the usual way (as in 3.13). $M^{\mathcal{U}}_{\gamma+1}$ is not a premouse, as $F^{\mathrm{Ult}} \restriction \kappa = F^{M|\beta} \restriction \nu_{M|\beta} \notin \mathrm{Ult}$, though κ is a generator of F^{Ult}. Also, ν_E is the sup of generators of F^{Ult} (as in 3.13), though $\nu_E < i_E(\kappa)$, the largest cardinal of Ult. So it fails the initial segment condition, and F^{Ult} is not its own trivial completion. Ult does satisfy the remaining premouse axioms: as in 3.13, applying F^{Ult} to M is equivalent to the two-step iteration starting with M, and applying E^M_β to form P, followed by E. Coherence of F^{Ult} follows.

If $\gamma+1 <_{\mathcal{U}} \delta$ and there is no drop from $\gamma+1$ to δ, the above situation generalizes to $M^{\mathcal{U}}_\delta$. In particular, if $E^{\mathcal{U}}_\delta$ is the active extender of $M^{\mathcal{U}}_\delta$, then it applies to M, and $\mathrm{Ult}_0(M, E^{\mathcal{U}}_\delta)$ is the model given by the iteration starting with P, and using the extenders

$$\{E^{\mathcal{U}}_{\gamma'} \mid \gamma+1 \leq_{\mathcal{U}} \gamma'+1 \leq_{\mathcal{U}} \delta\}. \tag{9}$$

In this case we'll say $M^{\mathcal{U}}_\delta$ is *anomalous*, and so is its active extender.

Around anomalous structures, we also need to tweak the rule dictating which model an extender applies to. Given E with $\mathrm{crit}(E) > \kappa$, E will apply to (the largest possible segment of) $M^{\mathcal{U}}_\delta$, where δ is least such that $\mathrm{crit}(E) < \nu^{\mathcal{U}}_\delta$, or $E^{\mathcal{U}}_\delta$ is anomalous, and $\mathrm{crit}(E) < i^{\mathcal{U}}_{M|\beta,\delta}(\kappa)$. (Anomalous extenders are active, so coherence ensures enough agreement between models for this. The rule also guarantees generators are never moved, as usual.)

Claim 1. *$\mathcal{P}$ is ω_1+1-iterable for iterations following the rules described above, and whose extenders are indexed above λ.*

Proof. Let $i_\mu(M) = \mathrm{Ult}(M, \mu)$ and

$$\mathcal{P}' = (i_\mu(M), \mathrm{Ult}(i_\mu(M), i_\mu(E^M_\beta)\restriction\kappa^+), \kappa^+).$$

Since $i_\mu(M)$ is iterable by assumption, $\mathcal{P}'$ is clearly iterable for iterations using extenders indexed above $(\kappa^{++})^{\mathrm{Ult}(i_\mu(M), i_\mu(E^M_\beta)\restriction\kappa^+)}$, the length of the trivial completion of $i_\mu(E^M_\beta)\restriction\kappa^+$. We will reduce the necessary iteration of $\mathcal{P}$ to such an iteration of $\mathcal{P}'$. We start with $\pi_0 = i_\mu : M \to i_\mu(M)$ and define $\pi_1 : \mathrm{Ult}(P, \bar{\mu}) \to \mathrm{Ult}(i_\mu(M), i_\mu(E^M_\beta)\restriction\kappa^+)$. First let $\pi : \mathrm{Ult}(P, \bar{\mu}) \to i_\mu(P)$ be the canonical factor map, which is 1-elementary. Now $i_\mu(P) = \mathrm{Ult}(i_\mu(M), i_\mu(E^M_\beta))$, and $\mathrm{rg}(\pi) = \mathrm{Hull}_1^{i_\mu(P)}(\kappa+1)$. Since the natural factor embedding

$$\mathrm{Ult}(i_\mu(M), i_\mu(E^M_\beta)\restriction\kappa^+) \to \mathrm{Ult}(i_\mu(M), i_\mu(E^M_\beta)) = i_\mu(P)$$

is the identity below κ^+, we get π_1. Note π_1 is 1-elementary, $\mathrm{crit}(\pi_1) = \beta$, and $\pi_1(\beta) = \kappa^+$. We are only considering iterations of $\mathcal{P}$ in which the first extender is indexed above $(\kappa^{++})^{\mathrm{Ult}(P,\bar{\mu})}$, so the lifted iteration begins with high enough index.

The copying process is standard except at anomalies. We're lifting $\mathcal{U}$ to $\mathcal{V}$. Say $E = E^{\mathcal{U}}_\alpha$ has crit κ, so it applies exactly to $M|\beta$. Then $F = E^{\mathcal{V}}_\alpha = \pi_\alpha(E^{\mathcal{U}}_\alpha)$ also has crit κ (since $\pi_1(\kappa) = \kappa$), so F is to be applied to $i_\mu(M)$. We'll define

$$\pi_{\alpha+1} : \mathrm{Ult}_0(M|\beta, E) \to i_F \circ i_\mu(M|\beta).$$

Since $\mathrm{crit}(\pi_0) = \kappa$ but $\mathrm{crit}(\pi_\alpha) > \kappa$, one needs to use the simple variation on the shift lemma employed in the proof of ([21], 6.11). I.e., define $\pi_{\alpha+1}$ by

$$[a, f] \mapsto [\pi_\alpha(a), i_\mu(f)\restriction\kappa].$$

Because all our copy maps fix points below κ, it's easy to see that the proof of the shift lemma still goes through with this definition ([21] has some details). $\pi_{\alpha+1}$ is a weak 0-embedding. (It seems likely that it won't be 1-elementary because of the extra functions used in forming

$i_F \circ i_\mu(M|\beta)$.) If $E^{\mathcal{U}}_{\alpha+1}$ is the active extender of $\mathrm{Ult}_0(M|\beta, E)$, then set $E^{\mathcal{V}}_{\alpha+1} = i_F \circ i_\mu(E^M_\beta)$ - clearly $\pi_{\alpha+1}$ then suffices for the shift lemma. By commutativity,

$$\pi_{\alpha+1}(i_E(\kappa)) = i_F \circ i_\mu(\kappa) = \nu^{\mathcal{V}}_{\alpha+1}.$$

Therefore if $\alpha + 1 < \gamma$ then

$$\mathrm{crit}(E^{\mathcal{U}}_\gamma) < i_E(\kappa) \iff \mathrm{crit}(E^{\mathcal{V}}_\gamma) < \nu^{\mathcal{V}}_{\alpha+1}.$$

So our rule for which model to return to in $\mathcal{U}$ lifts to the usual rule for $\mathcal{V}$. Finally, suppose $E^{\mathcal{U}}_\gamma$ is to return to an anomalous structure $M^{\mathcal{U}}_\delta$, without triggering a drop in $\mathcal{U}$. Then $\mathrm{crit}(E^{\mathcal{U}}_\gamma) < i^{\mathcal{U}}_{M|\beta,\delta}(\kappa)$, and since $\pi_\delta \circ i^{\mathcal{U}}_{M|\beta,\delta}(\kappa)$ is a cardinal of $M^{\mathcal{V}}_\delta$, there's also no drop triggered in $\mathcal{V}$ by $E^{\mathcal{V}}_\gamma$. So we define things at the $\gamma+1$ level as we did for $\alpha+1$ in the preceding (though using the standard shift lemma), and it works the same. □(Claim 1)

Armed with iterability, we complete the proof. Compare M with $\mathcal{P}$. Let $\mathcal{T}$ be the tree on M and $\mathcal{U}$ the tree on $\mathcal{P}$. All extenders used in $\mathcal{U}$ do have length above λ, since $\mathrm{Ult}(P, \bar{\mu})|\lambda = \mathrm{Ult}(M, E^M_\beta)||\lambda$. (So $E^{\mathcal{T}}_0 = E^M_\beta$.) Since $M = \mathrm{Th}_1(M)$ and the models in $\mathcal{P}$ have the same Σ_1 theory, $\mathcal{T}$ and $\mathcal{U}$ have the same final model Q and there is no dropping on either main branch. Suppose Q is above M in $\mathcal{U}$. Unless the branch begins with an anomalous extender, a contradiction is achieved by compatible extenders as usual.

Suppose $b^{\mathcal{U}}$'s first extender $F = E^{\mathcal{U}}_\delta$ is anomalous. We adopt the notation around (9); in particular, γ is least such that $\gamma + 1 \leq_{\mathcal{U}} \delta$. F's action on M is an iteration beginning with E^M_β and $E^{\mathcal{U}}_\gamma$. Since F's generators aren't moved along b, $P = \mathrm{Hull}^Q_1(\kappa)$, and the hull embedding from P to Q is compatible with $E^{\mathcal{U}}_\gamma$ through $\nu^{\mathcal{U}}_\gamma$. The usual arguments then give $E = E^M_\beta = E^{\mathcal{T}}_0$ is the first extender used on $b^{\mathcal{T}}$, resulting in P. Since $i^{\mathcal{T}}_{P,Q}$ has crit $\geq \kappa$, it agrees with the hull embedding, implying $E^{\mathcal{U}}_\gamma$ is compatible with the second extender used along $b^{\mathcal{T}}$.

So Q is above $U = \mathrm{Ult}(P, \bar{\mu})$. Since $\mathrm{crit}(\bar{\mu}) = \kappa$ and $\mathrm{crit}(i^{\mathcal{U}}_{U,Q}) > \kappa$, the argument of the last paragraph shows $b^{\mathcal{T}}$ uses E^M_β first and an extender compatible with $\bar{\mu}$ second, so that $\bar{\mu}$ is on the P-sequence. □(Theorem 4.15)

Remark 4.16. Suppose M is a sound mouse with active extender E, such that $E \restriction \gamma + 1$ is a type Z segment. The initial segment condition for premice doesn't appear to give $E \restriction \gamma + 1 \in M$, but ([20], 2.7) does (as we're assuming M is iterable). There is no proof of this theorem in [20]. The foregoing argument can easily be adapted to prove the following: let $\bar{\mu}$ be the normal measure over $\mathcal{P}(\gamma) \cap \mathrm{Ult}(M, E \restriction \gamma)$ given by the factor embedding

$$\mathrm{Ult}(M, E \restriction \gamma) \to \mathrm{Ult}(M, E \restriction \gamma + 1).$$

Then $\bar{\mu}$ is on the $\mathrm{Ult}(M, E \restriction \gamma)$ sequence. It's easy to see that $(\gamma^+)^{\mathrm{Ult}(M,E\restriction\gamma)}$ is the next generator of E after γ, so this gives $E \restriction \gamma + 1 \in M$.
For this, we may assume M has only one more generator after γ. Just compare M with the phalanx $(M, \mathrm{Ult}(M, E \restriction \gamma + 1), \gamma + 1)$. The iterability is obtained by reducing an iteration to one on $(M, \mathrm{Ult}(M, E), \nu_E)$. One faces the same complications as in the above proof.

5 Stacking Mice

Given a mouse M, one might ask whether $\mathbb{E}^M$ is definable without parameters over M's universe. If so, clearly $V = \text{HOD}$ in M. Steel showed that for $n \leq \omega$, the answer is "yes" for M_n. He actually showed M_n satisfies $\mathsf{V} = \mathsf{K}$ (and $\mathbb{E} = \mathbb{E}^K$) "between" consecutive Woodins. An argument is given in [17]. Although this works for mice higher in the mouse order, exactly how high seems unknown. In this section, we give a new proof of "yes" for M_n ($n \leq \omega$), and extend the result to various other mice. We argue without K, just using the following type of internal definition of $\mathbb{E}$: given $\mathbb{E}\restriction\kappa$ for some cardinal κ, $\mathbb{E}\restriction\kappa^+$ is obtained by stacking appropriate mice projecting to κ. (Clearly this is related to K, especially in light of Schindler's result in [**?**], that above $\aleph_2$, K is the stack of projecting mice.) For this definition to work, M must at least be sufficiently self-iterable. Not far into non-tame mice, our methods break down because of this requirement (see 5.16). We'll need to appeal to the argument of 3.7 to see that the iterability of candidate mice actually guarantees that they are initial segments of $\mathcal{J}^{\mathbb{E}}$. First we consider the type of iterability needed.

Definition 5.1. Let P be a sound premouse with a cardinal $\rho \leq \rho_\omega^P$. Then Σ is an α, ρ-*extender-full* iteration strategy for P if:

- Σ is an α-iteration strategy for P above ρ (meaning for trees above ρ).
- Whenever Q is the last model of an iteration via Σ, and E is such that $(Q||\text{lh}(E), E)$ is a premouse with $\text{crit}(E) < \rho$,

$$\text{Ult}_\omega(P, E) \text{ is wellfounded} \quad \Longleftrightarrow \quad E \text{ is on the } Q \text{ sequence.}$$

Let M be a premouse satisfying ZFC, and η a cardinal of M. M is *extender-full self-iterable at* η if for each $P \trianglelefteq M$ such that $\rho_\omega^P = \eta$,

$$M \models P \text{ is } (\eta^+ + 1), \eta\text{-extender-full iterable.}$$

Lemma 5.2. *Suppose M is a premouse satisfying* ZFC*, and M is extender-full self-iterable at η. Suppose $P \in M$ is a sound premouse extending $M|\eta$, projecting to η, and such that*

$$M \models P \textit{ is } (\eta^+ + 1), \eta\textit{-extender-full iterable.}$$

Then $P \trianglelefteq M$.

Therefore if M is extender-full self-iterable at all of its cardinals, $M \models V = \text{HOD}$.

Proof. We have $P \in M|\beta$ where $M|\beta$ projects to η. In M, we can compare P with $M|\beta$ using their extender-full iteration strategies. For suppose E is on the sequence of an iterate of P, with $\text{crit}(E) < \eta$, so by extender-fullness, $\text{Ult}_\omega(P, E)$ is wellfounded. Since η is a cardinal of M and P agrees with $M|\beta$ below η, E is total over M. Moreover, $\text{Ult}(M, E)$ is wellfounded, since $M|(\text{crit}(E)^+)^M = P|(\text{crit}(E)^+)^P$ (as in the 3rd paragraph of the proof of 2.9, witnesses to illfoundedness can be collapsed into $M|(\text{crit}(E)^+)^M$). So $\text{Ult}_\omega(M|\beta, E)$ is also wellfounded.

Therefore E can't be used during comparison. For doing so would require agreement at that stage below $\mathrm{lh}(E)$, so by extender-fullness, E would also be on the model above $M|\beta$, so it *doesn't* get used in comparison; contradiction. The same argument applies in the opposite direction, so only extenders with crit $\geq \eta$ need be used during comparison, so the iteration strategies suffice. Since the models being compared project to η and both sides of the comparison are above η, $P \lhd M|\beta$. □(Lemma 5.2)

Corollary 5.3. *For $n \leq \omega$, $M_n \models V = \mathrm{HOD}$.*

Proof. Let $\delta_0, \ldots, \delta_{n-1}$ be the Woodins of M_n, $\delta_{-1} = 0$. $M_n|\delta_k$ knows its own iteration strategy for trees above δ_{k-1} of length $< \delta_k$. (Given some tree $\mathcal{T}$, choose the branch b such that $\mathrm{Col}(\omega, \mathcal{T})$ forces $Q(b, \mathcal{T})$ to be Π^1_{n-k}-iterable. Here "Π^1_1-iterable" just means wellfounded. See the next section or [23] for discussion.) By 3.7 relativized to iteration strategies above some cut-point α (see the remark following 3.7), this strategy is δ_k, ρ-extender-full for each M_1-cardinal ρ in the interval $[\delta_{k-1}, \delta_k)$. Therefore 5.2 applies.

If δ_i is the i^{th} Woodin of M_ω, $M_\omega|\delta_i$ also knows its own strategy above δ_{i-1}. This follows from [([22], §7), but we provide an argument here. Consider a tree $\mathcal{T}$ on and in $M_\omega|\delta_0$. The correct branch b is that for which $Q(b, \mathcal{T})$ is weakly $(\deg^{\mathcal{T}}(b), \omega)$-iterable. This condition is $\Sigma_1^{L(\mathbb{R})}$. Since M_ω can compute $L(\mathbb{R})$ truth via the symmetric collapse of its Woodins, and the collapse is homogeneous, $b \in M_\omega$, and M_ω defines the correct strategy. But therefore in $M_\omega|\delta_0$, $Q(b, \mathcal{T})$ is also λ-iterable above $\delta(\mathcal{T})$, for any $\lambda < \delta_0$. Since $Q(b, \mathcal{T})$ has size $< \delta_0$, such iterability suffices to identify it. Now apply 3.7 (relativized above a cut-point) and 5.2. □(Corollary 5.3)

Remark 5.4. The self-iterability of M_ω established above is not optimal. For example, let $\alpha < \delta_0$; then using genericity iterations, one can see that $M_\omega|(\alpha^+)^{M_\omega}$ can also define its own strategy restricted to trees it contains (see ([22],§7)). The above argument only gives this at limit cardinals.

We now move on to more complex mice, where self-iterability is more difficult to establish. Given some $\mathcal{T}$, we'll build an appropriate fully backgrounded $L[\mathbb{E}]$ construction over $M(\mathcal{T})$ to search for its Q-structure. To prove this search is successful (the construction reaches the Q-structure), we need to restrict to mice which "rebuild themselves". That is, the mouse will be minimal for some hypothesis φ, and we'll establish that an unsuccessful $L[\mathbb{E}]$ construction leads to a model satisfying φ. We'll also establish that each stage of the construction is in fact an iterate of the Q-structure, above $\delta(\mathcal{T})$. The Q-structure will have no level modelling φ, so the construction *must* be successful.

The method employed for maintaining that the levels of a construction with good enough background certificates are iterates of some mouse was described to the author by Steel, and is an amalgamation of ideas from 3.2 and 3.3 of [16]. This is the core of the argument; our contribution was in noticing that these ideas lead to 5.11 and, combined with the argument for 3.7, to 5.14.

Definition 5.5. Let P be a sound premouse and $\lambda \leq \mathrm{OR} + 1$. A sequence $\langle N_\alpha \rangle_{\alpha < \lambda}$ is an *ms-array above* P if $N_0 = P$, $\mathrm{OR}^P \leq \rho_\omega^{N_\alpha}$ for $\alpha + 1 < \lambda$, and the sequence satisfies the requirements of a K^c construction (see [22]) *other than*

(a) $N_0 = V_\omega$, and

(b) The existence of background certificates.

If $P = V_\omega$, the sequence is simply an *ms-array*.

The following definition is a variation on that of a weak $\mathcal{A}$-certificate ([16], 2.1).

Definition 5.6. Suppose P is a premouse with active extender F. Let $\kappa = \text{crit}(F)$, $\nu = \nu_F$, and $x \in \mathbb{R}$. An *x-certificate (for P)* is an elementary $\pi : N \to V_\theta$ such that

(a) N is a transitive, power admissible,

$$\mathcal{P}(\kappa)^P \cup \{x\} \subseteq N,$$

(b) $F \restriction (P \times [\nu]^{<\omega}) = E_\pi \restriction (P \times [\nu]^{<\omega})$,

(c) $\pi(P|\kappa)||\text{lh}(F) = P||\text{lh}(F)$.

A premouse P with active extender F is *x-certified* if there is an x-certificate for P.

An *ω-mouse* is an ω-sound, $\omega_1 + 1$-iterable mouse projecting to ω.

An ms-array $\mathbb{C} = \langle N_\alpha \rangle_{\alpha<\lambda}$ is *mouse certified* if for each active $N_{\alpha+1}$: if there's an ω-mouse not in $N_{\alpha+1}$, and M is least such, then $N_{\alpha+1}$ is M-certified. $\mathbb{C}$ is *mouse maximal* if it's mouse certified, and $N_{\alpha+1}$ is active whenever possible under this constraint.

An ms-array $\langle N_\alpha \rangle_{\alpha<\lambda}$ *reaches* M if there is α such that $M = \mathfrak{C}_\omega(N_\alpha)$.

Remark 5.7. Having the codomain of an x-certificate π be a V_θ isn't that important; all we really need is that it is transitive and contains $V_{\omega+2}$.

Lemma 5.8. *Let $\langle N_\alpha \rangle_{\alpha<\lambda}$ be a mouse certified ms-array, where $\lambda \leq \omega_1$. Let M be an ω-mouse, and let Σ be M's unique $\omega_1^+ + 1$ strategy. Then for each α, either $M \trianglelefteq N_\alpha$, or there is a Σ-iterate P of M such that $N_\alpha \trianglelefteq P$. In other words, N_α does not move in Σ-comparison with M.*

Proof. The proof is like part of the proof of 5.14, so we omit it. □(Lemma 5.8)

Corollary 5.9. *Let $\mathbb{C}, \mathbb{C}'$ be countable mouse maximal constructions of the same length. If $\mathbb{C}$ does not reach some ω-mouse, then $\mathbb{C} = \mathbb{C}'$. Therefore there's a unique maximal construction of minimal (limit) length reaching all reachable ω-mice; denote this by $\mathbb{C}^*$.* □

Lemma 5.10. *Suppose $\mathbb{C}$ is a mouse maximal construction of length ω_1 and M is an ω-mouse. Then $\mathbb{C}$ reaches M. Thus if there are ω_1 many ω-mice, the output of $\mathbb{C}^*$ is the stack of them all.*

Proof. Assume otherwise, and let M be the least counterexample, with iteration strategy Σ. Let $W = N_{\omega_1}$ be the natural limit of the construction; notice W has height ω_1, and $W|\omega_1^W = M|\omega_1^M \lhd M$. By 5.8, $W \trianglelefteq M^{\mathcal{T}}_{\omega_1}$ for a Σ-iteration $\mathcal{T}$ of M of length $\omega_1 + 1$. Let θ have high cofinality and $V_\theta \prec_k V$ for some large k. Let $X \prec V_\theta$ be countable, transitive below ω_1, $\langle N_\alpha \rangle \in X$ and $M \in X$. Let $\pi : N \to V_\theta$ be the hull uncollapse of X. Let $\kappa = \text{crit}(\pi)$, so $\pi(\kappa) = \omega_1$. $\mathbb{C}$ is M-maximal after β, where $M|\omega_1^M = N_\beta$; we will show that π contradicts this maximality.

Let $b = \Sigma(\mathcal{T} \upharpoonright \omega_1)$; note $\pi(b \cap \kappa) = b \in X$. So b is unbounded below κ by elementarity, and as b is club in ω_1, we get $\kappa \in b$. Thus $b \cap \kappa = \Sigma(\mathcal{T} \upharpoonright \kappa) \in N$ and $\pi(\mathcal{T} \upharpoonright \kappa + 1) = \mathcal{T}$. In particular, $M^{\mathcal{T}}_\kappa \in N$.

$$\pi \upharpoonright M^{\mathcal{T}}_\kappa = i^{\mathcal{T}}_{\kappa,\omega_1}; \tag{10}$$

this is as in the proof that comparison terminates: for $\eta < \kappa$ and $x \in M^{\mathcal{T}}_\eta$, since η, x are countable in N,

$$\pi(i^{\mathcal{T}}_{\eta,\kappa}(x)) = i^{\mathcal{T}}_{\eta,\omega_1}(x) = i^{\mathcal{T}}_{\kappa,\omega_1} \circ i^{\mathcal{T}}_{\eta,\kappa}(x).$$

Since $\text{crit}(i^{\mathcal{T}}_{\kappa,\omega_1}) = \kappa$, $\mathcal{P}(\kappa) \cap W = \mathcal{P}(\kappa) \cap M^{\mathcal{T}}_\kappa$. Let $F = E_\pi \upharpoonright (W \times [\omega_1]^{<\omega})$. By (10), F is compatible with some $E^{\mathcal{T}}_\gamma$. ($E^{\mathcal{T}}_\gamma$ also measures exactly $\mathcal{P}(\kappa) \cap W$.) Now $\text{lh}(E^{\mathcal{T}}_\gamma) < \omega_1$ is a successor cardinal of W. So $W|\text{lh}(E^{\mathcal{T}}_\gamma) = N_\alpha$ where α is the limit of stages β so that N_β projects strictly below $\text{lh}(E^{\mathcal{T}}_\gamma)$. But π is an M-certificate for $(N_\alpha, E^{\mathcal{T}}_\gamma)$. So by mouse maximality, $N_{\alpha+1} = (N_\alpha, E)$ for some E, so $N_{\alpha+1}$ projects below $\text{lh}(E^{\mathcal{T}}_\gamma)$. Contradiction. □(Lemma 5.10)

Corollary 5.11. *All ω-mice are reachable. Therefore $\mathbb{C}^*$ (as in 5.9) reaches all ω-mice.*

Proof. Suppose M is an unreachable ω-mouse, with iteration strategy Σ. Let N be the last model of a mouse maximal construction $\mathbb{C}$. By 5.8, there's a Σ-iterate Q of M such that $N \trianglelefteq Q$. Therefore $\mathfrak{C}_\omega(N)$ exists, and $\mathbb{C}$ can be properly extended. Any ms-array of limit length can be uniquely properly extended. By 5.9, there is at most one mouse maximal construction of any given countable length. So we can build a unique mouse maximal construction of length ω_1. By 5.10, this construction reaches M at some countable stage; contradiction. □(Corollary 5.11)

Lemma 5.12. *Let $\langle N_\alpha \rangle$ be an ms-array, with last model N. Let E on $\mathbb{E}^N_+$ be total over N and such that ν_E is a cardinal of N. Then there is α such that $N_{\alpha+1} = (N_\alpha, E)$.*

Proof. Let E' be the original ancestor of E (so there is a stage β so that $N_{\beta+1} = (N_\beta, E')$ and E' eventually collapses to E during later stages of construction). Because ν_E is a cardinal of N, if $E' \neq E$ then E is the trivial completion of $E' \upharpoonright \nu_E$, and if $\gamma \geq \beta$ then $\rho^{N_\gamma}_\omega \geq \nu_E$. So ν_E is a cardinal of N_β and E is on the N_β sequence. Now apply induction with N_β and E. □(Lemma 5.12)

Definition 5.13. Suppose $V = L[\mathbb{E}]$. Let P be a sound premouse. An ms-array $\mathbb{C} = \langle N_\alpha \rangle$ above P is *appropriate* if whenever $N_{\alpha+1} = (N_\alpha, E)$, there is a total extender G such that

- G is indexed on $\mathbb{E}$,

- $\nu_G \geq \nu_E^+$ and is a cardinal,
- $G \restriction (N_\alpha \times [\nu_E]^{<\omega}) = E \restriction (N_\alpha \times [\nu_E]^{<\omega})$,
- $i_G(N_\alpha)||\mathrm{lh}(E) = N_\alpha$.

An appropriate ms-array is *maximal* if it adds an extender whenever possible. By minimizing the choice of background extenders, one obtains a canonical such construction.

Theorem 5.14 (Steel, Schlutzenberg)**.** *Let M be the least non-tame mouse. Then $M|\mathrm{crit}(F^M)$ is extender-full self-iterable at its cardinals, so satisfies $V = \mathrm{HOD}$.*

Proof. By 5.11 there is a countable mouse maximal construction $\langle N_\alpha \rangle_{\alpha \leq \xi+1}$ such that $M = \mathfrak{C}_\omega(N_{\xi+1})$. Note that with β such that $N_\beta = M|\omega_1^M$, for all $\alpha + 1 > \beta$, if $N_{\alpha+1}$ is active, then it's M-certified. Note $N_{\xi+1} = (N_\xi, E)$, N_ξ is tame, and $\delta = \nu_E$ is a Woodin cardinal in N_ξ. Let $N = N_\xi$ and $\kappa = \mathrm{crit}(E)$.

Work in N. Let $\eta < \kappa$ be a cardinal and $N|\beta$ project to η. Let Γ be the following partial strategy for trees on $N|\beta$ above η. Suppose $\mathcal{T}$ is of limit length via Γ. Let Q' be the Q-structure for $\delta(\mathcal{T})$ reached by the canonical appropriate ms-array above $M(\mathcal{T})$ through δ stages. Let Q be the δ-hull of Q': i.e. if $\rho_\omega^{Q'} \geq \delta$ then $Q = \mathfrak{C}_\omega(Q')$; otherwise $Q = \mathrm{Hull}_{n+1}^{\mathfrak{C}_n(Q')}(\delta)$ where $\rho_{n+1}^{Q'} < \delta \leq \rho_n^{Q'}$. Then $\Gamma(\mathcal{T})$ is the unique b such that $Q \trianglelefteq M_b^{\mathcal{T}}$. (If this definition fails or yields an illfounded branch, $\Gamma(\mathcal{T})$ is undefined.) We will show that Γ is a κ, η-extender-full strategy for $N|\beta$. Moreover (from outside N), Γ agrees with the strategy $\Sigma_{N|\beta}$ for $N|\beta$ inherited from M's strategy and 5.8.

Let $\mathcal{T}$ be of limit length via both Γ and $\Sigma_{N|\beta}$. Let $\langle P_\alpha \rangle$ be the models of N's canonical appropriate ms-array above $M(\mathcal{T})$ through κ stages, or until a Q-structure for $M(\mathcal{T})$ is reached. The construction doesn't break down before reaching a Q-structure, using Claim 5.8 and that M is iterable. In fact, let $Q(\mathcal{T})$ be the Σ-blessed Q-structure for $M(\mathcal{T})$. Then:

Claim 1. *For each α, P_α is a segment of a Σ-iterate of $Q(\mathcal{T})$ above $\delta(\mathcal{T})$ (and so a Σ-iterate of M).*

Proof. We proceed by induction on α. By Claim 5.8, $N|\beta$ is a segment of a Σ-iterate of M. Since $\mathcal{T}$ is via $\Sigma_{N|\beta}$ and $P_0 = M(\mathcal{T}) \trianglelefteq Q(\mathcal{T})$, the claim holds at $\alpha = 0$.

The case $\alpha = 0$ is trivial as $P_0 = M(\mathcal{T})$.

Assume $P_\alpha \trianglelefteq R$, where R is the last model of the tree $\mathcal{U}$ on $Q(\mathcal{T})$ above $\delta(\mathcal{T})$, and that P_α is not a Q-structure for $M(\mathcal{T})$.

If P_α is unsound, then $P_\alpha = R$ and $M(\mathcal{T}) \trianglelefteq \mathfrak{C}_\omega(P_\alpha) \triangleleft M_\gamma^{\mathcal{U}}$ for some γ. (There must be a drop in model on $\mathcal{U}$'s main branch as we haven't yet reached a Q-structure.) Therefore $P_{\alpha+1} = \mathcal{J}_1(\mathfrak{C}_\omega(P_\alpha)) \trianglelefteq M_\gamma^{\mathcal{U}}$, so $\mathcal{U} \restriction \gamma + 1$ works.

Suppose P_α is sound. Again here, $\mathcal{J}_1(P_\alpha) \trianglelefteq R$. So assume $P_{\alpha+1} = (P_\alpha, F)$. We must show F is the last extender used in $\mathcal{U}$. Let F^* be the canonical background extender for F in N. Then $\mu = \mathrm{crit}(F) = \mathrm{crit}(F^*)$ is not Woodin in P_α (by tameness of $P_\alpha|\mu$); let $P_\alpha|\gamma$ be the Q-structure for $P_\alpha|\mu$. Now ν_{F^*} is a cardinal of N. By 5.12 there is an M-certificate $\pi : S \to V_\theta$ for $N|\mathrm{lh}(F^*)$.

Since the construction $\langle N_\alpha \rangle$ reaching M was countable, $\mathrm{crit}(\pi) = \mu = \omega_1^S$. We have $\Sigma \in V_\theta$. Let Σ^S be S's (unique) μ+1-strategy for M. It's easy to see Σ^S actually agrees with Σ (where they both apply). (If $b = \Sigma^S(\mathcal{V})$ where $\mathcal{V}$ has length μ, then $\pi(b) \cap \mu = b$, so $b = \Sigma(\mathcal{V})$ also.) So by comparing M with $P_\alpha|\mu$, S obtains $\bar{\mathcal{U}} = \mathcal{U} \upharpoonright \mu + 1$ (with $M(\mathcal{U} \upharpoonright \mu) = P_\alpha|\mu$).

In V_θ, $\pi(\bar{\mathcal{U}})$ results from comparing M (via Σ) with $\pi(P_\alpha|\mu)$. Since $\mathrm{lh}(F) < \nu_{F^*}$ and $P_\alpha \in N$, 5.6 and 5.13 give

$$\pi(P_\alpha|\mu)||\mathrm{lh}(F) = i_{F^*}(P_\alpha|\mu)||\mathrm{lh}(F) = P_\alpha.$$

Since $\mathcal{U}$ is also via Σ, we get $\mathcal{U} \upharpoonright \lambda{+}1 = \pi(\bar{\mathcal{U}}) \upharpoonright \lambda{+}1$, where λ is least such that $\mathrm{lh}(E_\lambda^{\mathcal{U}}) \geq \mathrm{lh}(F)$ or $\mathrm{lh}(E_\lambda^{\pi(\bar{\mathcal{U}})}) \geq \mathrm{lh}(F)$.

Now as in (10) of 5.10, $M_\mu^{\mathcal{U}} = M_\mu^{\pi(\bar{\mathcal{U}})}$ and

$$i^{\pi(\bar{\mathcal{U}})}_{\mu,\pi(\mu)} = \pi \upharpoonright M_\mu^{\mathcal{U}}.$$

But $M_\mu^{\mathcal{U}} \cap \mathcal{P}(\mu) \supseteq P_\alpha \cap \mathcal{P}(\mu)$ and π, F^* and F agree on $P_\alpha \cap \mathcal{P}(\mu) \times [\nu_F]^{<\omega}$. So if $\xi + 1$ is least in $(\mu, \pi(\mu))_{\pi(\bar{\mathcal{U}})}$, then $E_\xi^{\pi(\bar{\mathcal{U}})}$ is compatible with F. The agreement between $\mathcal{U}$ and $\pi(\bar{\mathcal{U}})$ implies $\mathrm{lh}(E_\xi^{\pi(\bar{\mathcal{U}})}) \geq \mathrm{lh}(F)$. If F and $E_\xi^{\pi(\bar{\mathcal{U}})}$ measure the same sets, the initial segment condition implies F is on the $M_\xi^{\pi(\bar{\mathcal{U}})}$ sequence, and therefore on the $M_\lambda^{\pi(\bar{\mathcal{U}})} = M_\lambda^{\mathcal{U}}$ sequence as required. Otherwise F measures less sets, so $(\mu^+)^F < (\mu^+)^{E_\xi^{\pi(\bar{\mathcal{U}})}}$, which means F is type 1. (If $(\mu^+)^F < \nu_F$, then the identity of $(\mu^+)^F$ is coded directly into $F \upharpoonright \nu_F$). So 4.11 implies F is on the $M_\xi^{\pi(\bar{\mathcal{U}})}$ sequence in this case. (It seems plausible that this should arise when some normal measure E on μ is added to the construction, which remains total after constructing through all the ordinals. Condensation then gives cofinally many levels $\xi < (\mu^+)^E$ which are active with submeasures. Their original ancestors may have higher critical points, but if the F above was a submeasure's ancestor (with crit μ), then F measures less than E, but $E_\xi^{\pi(\bar{\mathcal{U}})}$ measures all of $M_\mu^{\mathcal{U}}$, which contains all sets measured by E.)

Finally consider a limit α. P_α is the lim inf of the agreeing segments of the sequence $\langle P_\beta \rangle_{\beta<\alpha}$. Let $\mathcal{U}_\beta$ witness the claim for P_β. Let $\mathcal{U}$ be the lim inf of the $\mathcal{U}_\beta$'s and let $M_\gamma^{\mathcal{U}}$ be its last model. If $M_\gamma^{\mathcal{U}}|\mathrm{OR}^{P_\alpha}$ is passive then $P_\alpha \trianglelefteq M_\gamma^{\mathcal{U}}$. Otherwise $\mathcal{U} \frown F$ works, where F is $M_\gamma^{\mathcal{U}}|\mathrm{OR}^{P_\alpha}$'s active extender: by induction, $\mathcal{U}$ is above $\delta(\mathcal{T})$, so $\delta(\mathcal{T}) < \mathrm{lh}(F)$. Since $\mathcal{U}$ is on $Q(\mathcal{T})$ and $Q(\mathcal{T})$ is tame (as $N|\beta$ is tame), $\mathrm{crit}(F) > \delta(\mathcal{T})$ also. Therefore $\mathcal{U} \frown F$ is above $\delta(\mathcal{T})$. □(Claim 1)

Now suppose a Q-structure is not reached and let $P = P_\kappa$. Let $\pi : S \to V_\theta$ be an M-certificate for E. (N, E) is an iterate of M, and $\pi(N|\kappa)$ is a segment of an iterate of M. By 5.6(c), $\pi(N|\kappa)||\mathrm{lh}(E) = N$, so E was used in the iteration producing $\pi(N|\kappa)$. So δ is Woodin in $\pi(N|\kappa)$ (as it's Woodin in $\mathrm{Ult}(N, E)$), and therefore also in $\pi(P)$. (Showing Woodinness goes into $\pi(P)$ is easy in our situation, using the same method we're about to use to show the strength of κ goes in.) Let $\mathcal{U}$ be the tree on M iterating out to P. As before, $\mathcal{U} \in S$ and has length $\kappa + 1$, and $\pi(\mathcal{U})$ is the tree iterating out to $\pi(P)$, using an extender compatible with

$$\pi \upharpoonright (\mathcal{P}(\kappa) \cap \pi(P)) \times [\delta]^{<\omega}.$$

As in the last paragraph of 5.10, that extender can't have length $\lambda < \delta$. Therefore it has natural length at least δ, so it is a non-tame extender. This contradicts Claim 1, since $Q(\mathcal{T})$ is tame.

So the construction does reach a Q-structure Q', and by Claim 1, its δ-hull Q (described at the start of this proof) is $Q(\mathcal{T})$. Since $\Sigma(\mathcal{T})$ is the unique branch b so that $Q(\mathcal{T}) \trianglelefteq M_b^{\mathcal{T}}$, we get that $\Gamma(\mathcal{T}) = \Sigma(\mathcal{T})$, as desired.

The facts established so far reflect down to M (the sound version of N), so (confusing notation a little) let κ,η,β,Γ now play the analagous roles in M that they did in N above (so κ is the critical point of M's active extender, etc.). It remains to show that Γ is a κ, η-extender-full strategy in M. So suppose $\mathcal{T}$ is a normal tree on $M|\beta$, above η, via Γ, of length $< \kappa$, and F fits on the sequence of $\mathcal{T}$'s last model, $\mathrm{crit}(F) < \eta$ and $\mathrm{Ult}_\omega(N|\beta, E)$ is wellfounded. As usual, since η is a cardinal of M, the wellfoundedness of $\mathrm{Ult}(M, E)$ follows. One can run the argument of 3.7, with a small alteration. Note that $\mathcal{T}$ is actually via Σ. For it is guided by Q-structures whose iterability is guaranteed by the fact that they are built by full background extender constructions inside M (or alternatively, by the fact that they lift to the correct Q-structures found inside N). By tameness, the correct branch is chosen, so Γ agrees with Σ. So the phalanx of Claim 1 of 3.7 will be $\omega_1 + 1$-iterable in V, since iterations reduce to iterations of the phalanx $\Phi(\mathcal{T})$, which is $\omega_1 + 1$-iterable in V. So we get E is on the sequence of $\mathcal{T}$'s last model, as desired. □(Theorem 5.14)

Remark 5.15. A similar argument (but simpler toward the end) can be used to show that other tame mice, such as the least mouse with an inaccessible limit of Woodins, are also extender-full self-iterable at their cardinals.

However, non-tame mice quickly produce situations where our method for proving 5.14 can't work. (Note that the self-iteration strategy obtained in 5.14 agreed with V's strategy.) Moreover, the approach used in proving 2.19 also fails in the following example, which is a simple variation on an observation possibly due to Steel, given in ([19], 1.1).

Fact 5.16. *Suppose N is a countable mouse modelling* ZFC*, τ is an N-cardinal, N has a cut-point η with $\tau < \eta < (\tau^+)^N$, and $P' \trianglelefteq P'' \triangleleft N$ are such that P' is active with E', $\mathrm{crit}(E') < \tau$, and $\delta > \tau$ is Woodin in P', and P'' projects to τ. Let Σ be an iteration strategy for N. Then N does not know Σ restricted to trees on P'', above τ, of length $\leq (\eta^+)^N$.*

Proof. Let $P \trianglelefteq N$ be least extending P' projecting strictly less than δ. Let $\rho_{n+1}^P < \delta \leq \rho_n^P$. Let $\kappa > \rho_{n+1}^{P'}$ be least that's measurable in P'. Let $\mathbb{P}$ be the extender algebra of P' with δ generators, using only crits above κ. In N, iterate P, first linearly with a normal measure on κ and its images η times, then iterate $M_\eta^{\mathcal{T}}$ to make $N|\eta$ generic over $i_{0,\xi}^{\mathcal{T}}(\mathbb{P})$, where $M_\xi^{\mathcal{T}}$ is $\mathcal{T}$'s last model. Let $E = i_{0,\xi}(E')$. Note $N|\eta$ is $\mathrm{Ult}(N, E)$ generic over $i_{0,\xi}^{\mathcal{T}}(\mathbb{P})$. Let $Q \trianglelefteq N$ be least projecting to τ, with $i_{0,\xi}^{\mathcal{T}}(\delta) \leq \mathrm{OR}^Q$. Since $\tau < i_{0,\xi}^{\mathcal{T}}(\delta)$ and the latter is a cardinal in $\mathrm{Ult}(N, E)[N|\eta]$, $Q \notin \mathrm{Ult}(N, E)[N|\eta]$. Let G be $\mathrm{Ult}(N, E)[N|\eta]$-generic for the collapse of OR^Q. In $\mathrm{Ult}(N, E)[N|\eta][G]$, let S be the tree of attempts to build a sound premouse R looking like Q: it should extend $N|\eta$, have η a cut-point, be sound and project to τ; S also builds an elementary $\sigma : R \to i_E(Q)$. S is illfounded because of Q and $i_E \restriction Q$; therefore $\mathrm{Ult}(N, E)[N|\eta][G]$ has such an R, σ. But clearly $i_E(Q)$ is iterable above $i_E(\eta)$, so

R is iterable above η, and it follows that $R = Q$. This was independent of the particular G, so $Q \in \mathrm{Ult}(N, E)[N|\eta]$; contradiction. $\square$(Fact 5.16)

However, neat non-tame mice might satisfy $\mathsf{V} = \mathsf{HOD}$ some other way. $M_{\mathsf{AD}_\mathbb{R}}$ is the minimal proper class mouse with a limit λ of Woodins which is a limit of cardinals strong below λ.

Question 5.17 (Steel)**.** *Does $M_{\mathsf{AD}_\mathbb{R}}$ satisfy $\mathsf{V} = \mathsf{HOD}$?*

Question 5.18 (Steel)**.** *Suppose M is a mouse modelling ZFC. Does M satisfy $\mathsf{V} = \mathsf{HOD}(X)$ for some $X \subseteq \omega_1^M$?*

A contender for the set X here is the set of all countable elementary substructures of levels of M. (This was used in Woodin's portion of the proof of 2.19.)

6 Homogeneously Suslin Sets

Kunen's analysis of the measures in $L[U]$ lead to the following observation of Steel:

$$L[U] \models \text{The homogeneously Suslin sets of reals are the } \mathbf{\Pi}^1_1 \text{ sets.}$$

Here we consider the situation in M_n ($n < \omega$), and in certain models below M_1. We establish in M_n an upper bound on the homogeneously Suslin sets a little below $\mathbf{\Delta}^1_{n+1}$. Certainly all $\mathbf{\Pi}^1_n$ sets there are homogeneously Suslin (by [7]), but we don't see how to improve either bound beyond this. We also show that in mice not too far above $0^\P$ and modelling ZFC, all homogeneously Suslin sets are $\mathbf{\Pi}^1_1$.

We first define the phrase "a little below".

Definition 6.1. Let N be an inner model of ZFC, which is $\mathbf{\Sigma}^1_{n+1}$-correct. Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be a standard universal $\mathbf{\Sigma}^1_{n+1}$ set, and for $z \in \mathbb{R}$ let U_z be the section of U at z. Suppose $A \subseteq \mathbb{R}$ is a $\mathbf{\Delta}^1_{n+1}$ set in N. Then A is (N)-*correctly*-$\mathbf{\Delta}^1_{n+1}$ iff there are $a, b \in \mathbb{R}^N$ so that $A = U_a \cap N$ and $U_a = \mathbb{R} - U_b$.

The definition is made in V, so N might not know which sets are correctly-$\mathbf{\Delta}^1_{n+1}$. But given $a, b \in \mathbb{R}^N$, whether (a, b) witnesses U_a is correctly-$\mathbf{\Delta}^1_{n+1}$ is a $\Pi^1_{n+2}(a, b)$ question. Since M_n can compute Π^1_{n+2} truth (of reals in M_n) by consulting the extender algebra (see [22]), we get: the class of M_n-correctly $\mathbf{\Delta}^1_{n+1}$ sets is definable over M_n.

Lemma 6.2. *Let δ_0 be the least Woodin of M_n. The M_n-correctly $\mathbf{\Delta}^1_{n+1}$ sets are precisely the the* $\mathrm{Col}(\omega, \delta_0)$*-universally Baire sets of M_n.*

Theorem 6.3. *In M_n, every homogeneously Suslin set of reals is correctly $\mathbf{\Delta}^1_{n+1}$; equivalently, they are* $\mathrm{Col}(\omega, \delta_0)$*-universally Baire, where δ_0 is the least Woodin cardinal.*

Corollary 6.4. *In M_n, the weakly homogeneously Suslin sets of reals are precisely the $\mathbf{\Sigma}^1_{n+1}$ sets.*

Proof. One direction follows the theorem and the fact that weakly homogeneously Suslin sets are just projections of homogeneously Suslin sets (see [6]). The other follows the Martin-Steel result of [7]. □(Corollary 6.4)

We expect that throughout the interval of mice from $0^\P$ to M_1, the homogeneously Suslin sets become steadily more complex descriptive set theoretically, culminating in:

Conjecture 6.5. *In M_1, the homogeneously Suslin sets, the correctly $\mathbf{\Delta}^1_2$ sets and the* $\mathrm{Col}(\omega, \delta_0)$ *sets coincide.*

Other related results have been known for some time. One was the following result of Woodin, which is discussed in [2]:

Fact 6.6 (Woodin)**.** *Assume* AD + DC*, and suppose there exists a normal fine measure μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$. For measure one many $\sigma \in \mu$, if g is a generic enumeration of $\mathbb{R} \cap M_1(\sigma)$ in order-type $\omega_1^{M_1(\sigma)}$, then the weakly homogeneously Suslin sets of $M_1(\sigma)[g]$ coincide with the $\mathbf{\Sigma}^1_2$ sets of $M_1(\sigma)[g]$.*

The analogous statement about $M_n(\sigma)$ and $\mathbf{\Sigma}^1_{n+1}$ sets also holds. Lower in the mouse order, Schindler and Koepke generalized the fact about $L[U]$ in [18] in a couple of ways. They showed that either if 0^{long} does not exist, or if $0^{\P}$ does not exist, $V = K$ and K is below a μ-measurable, then all homogeneously Suslin sets are $\mathbf{\Pi}^1_1$. In fact they make do with a notion weaker than homogeneously Suslin. (0^{long} and a μ-measurable are both well below $0^{\P}$. See 6.13 for a definition of $0^{\P}$.)

We now move on the proof of 6.2, and then 6.3. However, the proof of the theorem shows independently that the homogeneously Suslin sets are correctly $\mathbf{\Delta}^1_{n+1}$, then just quotes the lemma for universal Baireness. So the reader can skip the proof of the lemma if desired.

The following is established in ([23], 4.6).

Fact 6.7 (Woodin)**.** *Let $n \in \omega$, N be iterable and active, $\mathbb{P} \in N|\alpha$, and suppose N has n Woodin cardinals above α. Let G be $\mathbb{P}$-generic over N. Then $N[G]$ is $\mathbf{\Sigma}^1_{n+1}$-correct, and if n is even, $N[G]$ is $\mathbf{\Sigma}^1_{n+2}$-correct.* □

Corollary 6.8. *For n even, $M_n \models A$ is $\mathbf{\Delta}^1_{n+1}$ iff A is M_n-correctly $\mathbf{\Delta}^1_{n+1}$.*

Proof. Apply 6.7 to $M_n^\#$ and the $\Pi^1_{n+1}(z)$ statement "A is a $\Delta^1_{n+1}(z)$ set".

□(Corollary 6.8)

Remark 6.9. The proof we give for the " $\Longrightarrow$ " direction of the following lemma (in the $n > 1$ case) was provided by Steel. Our original attempt for the $n > 1$ case (which we sketch below) didn't work at all until Grigor Sargsyan pointed out the existence of 6.7, which makes it work for n odd. Thanks to both for the guidance.

Proof of Lemma 6.2. Consider first M_1. If $A \in M_1$ is correctly-$\mathbf{\Delta}^1_2$, use Shoenfield trees for A and its complement to witness the $\text{Col}(\omega, \delta_0)$-universal Baireness of A in M_1. Conversely, suppose

$$M_1|\eta \models \mathsf{ZF}^- \ + \ \text{Col}(\omega, \delta_0) \models S, T \text{ project to complements.}$$

We may assume S, T are definable points in $M_1|\eta$. Let $M = \text{Hull}_\omega^{M_1|\eta}(\emptyset)$ and let S^M, T^M be the collapses of S, T. Note that in iterating M, there is always a unique wellfounded branch (using 1-smallness and that $M = \text{Th}(M)$). Now define A by

$$x \in A \iff$$

$$\text{there's a non-dropping normal iterate } P \text{ of } M \text{ such that } x \in p[i_{M,P}(S^M)];$$

then

$$x \in \mathbb{R} - A \iff$$

$$\text{there's a non-dropping normal iterate } P \text{ of } M \text{ such that } x \in p[i_{M,P}(T^M)].$$

This is standard: given $x \in \mathbb{R}$, there is a non-dropping iterate of M so that x is generic over $\text{Col}(\omega, \delta_0)^P$ (see ([22], §7)), and $x \in p[i_{M,P}(S^M)]$ or $x \in p[i_{M,P}(T^M)]$ because these trees project to complements in $P[x]$. If there are iterates P_1 and P_2 so that $x \in p[i_{M,P_1}(S^M)]$ and $x \in p[i_{M,P_2}(T^M)]$, then let P be the result of coiterating P_1 and P_2. Since $M = \text{Th}(M)$, we get the same final model on both sides and $i_{P_1,P} \circ i_{M,P_1} = i_{M,P} = i_{P_2,P} \circ i_{M,P_2}$. But then $x \in p[i_{M,P}(S)] \cap p[i_{M,P}(T)]$, so by absoluteness, there is some such $x' \in P$, a contradiction. Since the formulas above are $\Sigma^1_2(M)$, the lemma holds when $n = 1$.

Now let $n > 1$. To show all $\mathrm{Col}(\omega, \delta_0)$-universally Baire sets are simple, the same proof as for $n = 1$ works, except that the extra complexity of the iteration strategy for the hull M leads only to a $\Delta^1_{n+1}(M)$ definition. (See [23], where the definability of the iteration strategy is discussed, or 6.3 for the $n = 2$ case.)

For the converse, consider first M_2 and (correctly) $\boldsymbol{\Delta}^1_3$. If G is generic for $\mathrm{Col}(\omega, \delta_0)$, $M_2[G]$ isn't Π^1_3-correct, but it can still compute Π^1_3 truth with its remaining extender algebra. This leads to a tree $T \in M_2$ which projects to $(\Pi^1_3)^V \cap M_2[G]$. For this, let $\varphi(v_3) = \forall v_1 \exists v_2 \psi(v_1, v_2, v_3)$ define a universal Π^1_3 set, with ψ Π^1_1. Fix $\eta > \delta_1$ such that $M_2|\eta \models \mathsf{ZF}^-$. Let T be the tree building (x, g, N, π), where $\pi : N \to M_2|\eta$ is elementary, g is N-generic over $\pi^{-1}(\mathrm{Col}(\omega, \delta_0))$, $x \in N[g]$, and

$$N[g] \models \text{The extender algebra at } \pi^{-1}(\delta_1) \text{ forces } \varphi(x).$$

Using genericity iterations, the reader can check that

$$p[T] \cap M_2[G] = \{x \in \mathbb{R} \cap M_2[G] \mid \varphi(x)\}.$$

(Notice that the corresponding tree S for Σ^1_3 doesn't work, since a Π^1_2 statement true in some $N[g]$ may be false in V.)

Now if $A \subseteq \mathbb{R}$ in M_2 is (correctly) $\boldsymbol{\Delta}^1_3$, then using T and $\boldsymbol{\Pi}^1_3$ definitions for A and its complement, it follows that A is $\mathrm{Col}(\omega, \delta_0)$-universally Baire in M_2.

In M_3, one obtains trees for Σ^1_4, etc. This completes the proof. □(Lemma 6.2)

Remark 6.10. We sketch another proof of the above lemma for n odd. Work in M_3. Let S_1 be a homogeneous tree for Π^1_2 obtained as in [7], with completeness κ, $\delta_0 < \kappa$. Let S_2 be the natural weakly homogeneous tree for Σ^1_3 obtained from S_1, and S_3 the corresponding Martin Solovay tree for Π^1_3, and S_4 the natural tree for Σ^1_4 obtained from S_3 (see [6] for details). Then since the completeness of all measures used is above δ_0, S_4 is still a tree for Σ^1_4 in $M_2[G]$, where G collapses δ_0. Given some M_3-correctly $\boldsymbol{\Delta}^1_4$ definition, let T_1 and T_2 be the corresponding slices of S_4. The definition extends to a $\boldsymbol{\Delta}^1_4$ set in $M_3[G]$, since $M_3[G]$ is Σ^1_4-correct, by 6.7. Therefore T_1 and T_2 project to complements in $M_3[G]$. This proof doesn't go through when n is even because $M_n[G]$ isn't sufficiently correct. But notice the resulting tree projects to $(\Sigma^1_{n+1})^{M_n[G]}$ in $M_n[G]$, whereas the tree used in the proof of 6.2 projects to $(\Pi^1_{n+1})^V \cap M_n[G]$. This gives another proof of the failure of Σ^1_{n+1} correctness in $M_n[G]$: if these sets are complementary, then every $\boldsymbol{\Sigma}^1_{n+1}$ set is δ_0-universally Baire in M_n, but then by 6.2, they're $\boldsymbol{\Delta}^1_{n+1}$; contradiction.

Proof of Theorem 6.3. First consider M_1, where we now work. Let δ be (the) Woodin. Suppose $T \in M_1$ is a homogeneous tree, definable as a point in $N = M_1|\eta$, where η is a double successor cardinal in M_1 above δ_0. Let $M = \mathrm{Hull}^N(\emptyset)$. We will give a canonical representation of $p[T]$ as a correctly-$\Delta^1_2(M)$ set. This implies the theorem, since the least homogeneous tree whose projection is not correctly-$\boldsymbol{\Delta}^1_2$ is definable.

Let $\pi : M \to N$ be elementary. M is $(\omega_1 + 1)$-iterable since N is. Moreover, because M is 1-small and pointwise definable, its strategy must always choose the unique wellfounded branch. Let $\langle \mu_s \rangle_{s \in {}^{<\omega}\omega} \in \text{rg}(\pi)$ be a homogeneity system for S. For $x \in {}^{\omega}\omega$, let

$$\bar{U}_x = \text{dirlim}_{m \leq n < \omega}(\text{Ult}(M, \bar{\mu}_{x \restriction n}), i^M_{x \restriction m, x \restriction n}),$$

where bars denote preimages under π.

Definition 6.11. For this proof, a countable premouse P is Π^1_2-*iterable* iff for every $\alpha, \mathcal{T}, x \in \mathbb{R}$ such that α codes an ordinal and $\mathcal{T}$ codes an iteration tree on U, either (a) there is a T-maximal branch b, cofinal in λ, such that $M^{\mathcal{T}}_b$ and $M^{\mathcal{T}}_\gamma$, $\gamma < \lambda$, are wellfounded through α; or (b) $\mathcal{T}$ has a final model, x codes a normal one step extension $\mathcal{T}'$ of $\mathcal{T}$, and all models of $\mathcal{T}'$ are wellfounded through α.

Claim 1. *For $x \in {}^{\omega}\omega$, the following are equivalent:*

(1) $x \in p[T]$;

(2) There is an elementary $\psi : \bar{U}_x \to M_1 | \eta$;

(3) $\bar{U}_x$ is Π^1_2-iterable;

(4) There is a countable Σ-iteration of M to a model Q, and an elementary $\sigma : \bar{U}_x \to Q$.

Moreover, the condition in (4) is $\Sigma^1_2(M)$.

Proof.
(1) $\implies$ (2).

Let U_x be the (wellfounded) direct limit in the S-system, with base model N. $\bar{U}_x \in U_x$, and clearly π yields an elementary $\pi_x : \bar{U}_x \to U_x$. Let $j : N \to U_x$ be the direct limit map. We can choose $\alpha < \delta$ such that $N|\alpha \preccurlyeq N$ and $\mathrm{rg}(\pi_x) \subseteq U_x|j(\alpha)$. By absoluteness, there is a $\psi' : \bar{U}_x \to U_x|j(\alpha)$ with $\psi' \in U_x$. Since $j(\bar{U}_x) = \bar{U}_x$, there's a $\psi : \bar{U}_x \to N|\alpha$.

(2) $\implies$ (3).

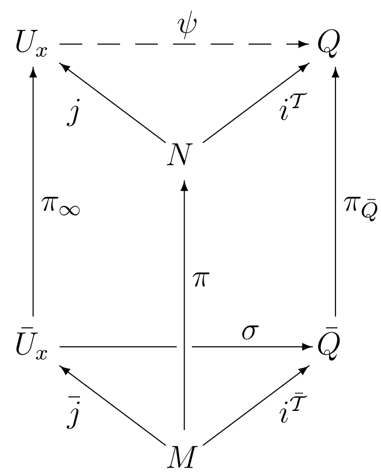
Immediate.

(3) $\implies$ (4).

As in [23], the Π^1_2-iterability of $\bar{U}_x$ allows us to run a comparison with M: whilst $\mathcal{T}$ on M chooses non-maximal branches, they provide Q-structures, which guide the branch choice for the tree $\mathcal{U}$ on $\bar{U}_x$. If $\mathcal{T}$ chooses a maximal branch, then $M(\mathcal{T})$ has the same theory as M and $\bar{U}_x$, so the Π^1_2-iterability also provides a maximal branch for $\mathcal{U}$. Note that if the comparison reaches such a maximal stage, then it has finished. The comparison cannot run for ω_1 stages, since otherwise it can be replicated in $L[M, \bar{U}_x]$, which has a smaller ω_1. So it terminates successfully, and since $M = \mathrm{Hull}^M(\emptyset)$, the same final model is produced by both trees, with no dropping on the main branches.

(4) $\implies$ (1).

Let $\bar{\mathcal{T}}$ be the Σ-tree on M, with final model $\bar{Q}$. Let $\mathcal{T} = \pi\bar{\mathcal{T}}$ be the copied tree on N and $\pi_{\bar{Q}}$ be the final copy map. Let $j : N \to U_x$, $\bar{j} : M \to \bar{U}_x$, $\pi_\infty : \bar{U}_x \to U_x$ be the natural maps. Then (ignoring ψ) the following diagram commutes:



We are missing one edge in this triagonal prism, which we need to complete the proof. We want to define $\psi : U_x \to Q$ in the only commuting way:

$$\psi(j(f)(\pi_\infty(b))) = i^{\mathcal{T}}(f)(\pi_{\bar{Q}} \circ \sigma(b)). \tag{11}$$

(All elements of U_x are of the form $j(f)(\pi_\infty(b))$ since the generators a' of $\mu_{x\restriction n}$ are in the range of π, so π_∞ hits $j_{n,\infty}(a')$.)

We need to see that this is well-defined and elementary. This requires certain measures derived from j and $i^{\mathcal{T}}$ to be identical.

Notation. Let $\mathcal{W}$ be a normal iteration tree on a premouse P with last model R, such that $i^{\mathcal{W}}$ exists. Let $x \in \mathrm{OR}^{<\omega}$, and suppose $x \in (i^{\mathcal{W}}(\kappa))^{<\omega}$, and that κ is in fact least such. Then $\mu^{\mathcal{W}}_x$ denotes the measure on $\kappa^{|x|}$ derived from $i^{\mathcal{W}}$ with generator x.

We may assume that in (11), $b = \bar{j}_{n,\infty}(a)$, where a is the generator of $\bar{\mu}_{x\restriction n}$. So $\pi(a) = a'$ where a' is as above. Now since the bottom triangle commutes, $\bar{\mu}_{x\restriction n} = \mu^{i^{\bar{\mathcal{T}}}}_{\sigma(b)}$. Let $\bar{\mathcal{T}}_{\sigma(b)}$ be the a finite support tree for $\sigma(b)$ derived from $\bar{\mathcal{T}}$ and $\bar{\tau} : \bar{R} \to \bar{Q}$ be the final copy map. Then since $\bar{\tau} \circ i^{\bar{\mathcal{T}}_{\sigma(b)}} = i^{\bar{\mathcal{T}}}$, $\bar{\mu}_{x\restriction n} = \mu^{\bar{\mathcal{T}}_{\sigma(b)}}_{\bar{\tau}^{-1}(\sigma(b))}$. Since $\bar{\mathcal{T}}_{\sigma(b)}$ is a finite tree on M, it is in M. Moreover, $\pi(\bar{\mathcal{T}}_{\sigma(b)}) = \pi\bar{\mathcal{T}}_{\sigma(b)}$ (the copied tree). Also, it's easy to see that $\pi\bar{\mathcal{T}}_{\sigma(b)}$ is a finite support tree for $\pi_{\bar{Q}}(\sigma(b))$ derived from $\mathcal{T}$. (Whilst extracting a support for $\sigma(b)$ from $\bar{\mathcal{T}}$, maintain inductively that the copy maps π_α lift it to a support for $\pi_{\bar{Q}}(\sigma(b))$; then the derived $\mathcal{T}_{\pi_{\bar{Q}}(\sigma(b))}$ is just $\pi\bar{\mathcal{T}}_{\sigma(b)}$.) Therefore

$$\mu^{\mathcal{T}}_{\pi_{\bar{Q}}\circ\sigma(b)} = \mu^{\mathcal{T}_{\pi_{\bar{Q}}(\sigma(b))}}_{\tau^{-1}(\pi_{\bar{Q}}(\sigma(b)))} = \pi(\mu^{\bar{\mathcal{T}}_{\sigma(b)}}_{\bar{\tau}^{-1}(\sigma(b))}) = \pi(\bar{\mu}_{x\restriction n}) = \mu_{x\restriction n}.$$

□(Claim 1)

The definability of (4) follows since a Σ-iterate of M is one which chooses wellfounded branches. This completes the proof of 6.3 for M_1.

Now consider M_2. The argument is almost as for M_1, with appropriate modifications to the conditions (1) - (4). The difference lies in the increased complexity of the iteration strategies for the hull M and the ultrapowers $\bar{U}_x$. Instead of Π^1_2-iterability, we need Π^1_3-iterability, which we presently define. This notion is also taken from [23].

Consider a 2-small, ω-sound mouse P projecting to ω. Its unique strategy, having built a limit length tree $\mathcal{T}$, must choose the unique branch b such that $Q(b, \mathcal{T})$ is Π^1_2-iterable above $\delta(\mathcal{T})$. As in the proof of (3) $\implies$ (4) above, such a $Q(b, \mathcal{T})$ can be compared to the Q-structure of the correct branch, so standard arguments show they're identical, and since P is sound, therefore so are the branches. This implies the statement "$\mathcal{T}$ is a correct normal tree on P" is $\Pi^1_2(P)$. Moreover, the correct branch is Δ^1_3 in any real coding $\mathcal{T}$. Assuming $\boldsymbol{\Delta}^1_2$-determinacy (true in M_2), $\Pi^1_3(x)$ is closed under "$\exists b \in \Delta^1_3(x)$" (see ([12], 4D.3, 6B.1, 6B.2)). This leads to the following formulation:

Definition 6.12. Assume $\boldsymbol{\Delta}^1_2$-determinacy. Let P be a 2-small, ω-sound premouse P projecting to ω. P is Π^1_3-*iterable* iff for each countable normal tree $\mathcal{T}$ on P, either $\mathcal{T}$ has a last model and every one-step normal extension produces a wellfounded next model, or there is a maximal branch b of $\mathcal{T}$ in $\Delta^1_3(\mathcal{T})$ such that $Q(b, \mathcal{T})$ is Π^1_2-iterable above $\delta(\mathcal{T})$.

With reals coding the elements of HC in this definition, the determinacy implies Π^1_3-iterability is indeed a Π^1_3-condition.

Now in M_2, the hull M is defined as before (in particular, it embeds in a level $M_2|\eta$ containing all Woodin cardinals). Although M doesn't literally project to ω, by 2.13 we can instead work with $\mathcal{J}_1(M)$, which is also sound by the same lemma. Conditions (1), (2) and (4) are as in the $n = 1$ case (with M_2 replacing M_1). For (4) though, we must define Σ. Since M_2 satsisfies "I'm δ_0-iterable", its unique $\omega_1 + 1$-iteration strategy for M is the pullback of its strategy for $M_2|\eta$. Σ denotes this strategy for M. The discussion above shows that condition (4) is then $\Sigma^1_3(M)$.

Condition (3) becomes "$\bar{U}_x$ is Π^1_3-iterable". However, $\mathcal{J}_1(\bar{U}_x)$ *isn't* sound in general, so we need to check that Π^1_3-iterability works in this context. (John Steel pointed out that it does in fact work, thereby simplifying our original argument, which instead used "Π^1_3-M-comparability".) Assume that $\bar{U}_x$ is fully iterable, via some Γ, and $\mathcal{T}$ is a normal tree of limit length on $\mathcal{J}_1(\bar{U}_x)$, with $\mathcal{T}\,^\frown b$ via Γ. At issue is the definability of b from $\mathcal{T}$; we need to see that b is the unique b' such that $Q(b', \mathcal{T})$ is Π^1_2-iterable above $\delta(\mathcal{T})$. The reader can check that things work as in the sound case unless b does not drop, and $i^{\mathcal{T}}_b(\delta^x_0) = \delta(\mathcal{T})$, where δ^x_0 is the least Woodin of $\bar{U}_x$. In this case, $Q(b, \mathcal{T}) = M^{\mathcal{T}}_b$, and is $\omega_1 + 1$-iterable above $\delta(\mathcal{T})$. Suppose $c \neq b$ and $Q(c, \mathcal{T})$ is Π^1_2-iterable above $\delta(\mathcal{T})$. As in the sound case, $Q(b, \mathcal{T})$ and $Q(c, \mathcal{T})$ can be successfully compared, and since they're Q-structures for $M(\mathcal{T})$, they iterate to the same model Q, with no dropping, and there was no drop along c. Let $j : \mathcal{J}_1(M) \to Q$ be the canonical embedding, which is continuous at δ^M_0 since it's composed of ultrapower embeddings of degree 0. Since M is pointwise definable, cofinally many points below $\delta(\mathcal{T})$ are definable in $Q|j(\mathrm{OR}^M)$. But these points are included in $\mathrm{rg}(i^{\mathcal{T}}_b) \cap \mathrm{rg}(i^{\mathcal{T}}_c)$, a contradiction (as in ([22], §6)).

Using similar arguments, one can show that if $\bar{U}_x$ is Π^1_3-iterable, then $\bar{U}_x$ is embeddable in a correct iterate of M. (One must compare 3 Q-structures simultaneously to see that during such comparison, the branches chosen by Π^1_3-iterability are always cofinal in the tree on $\bar{U}_x$.) Moreover, the comparison can be executed in M_2. For if the comparison ran through $\omega_1^{M_2}$ stages, then it could be replicated in $M_1(M, \bar{U}_x)$, using the extender algebra of that model to compute the correct branches. But then it runs through $\omega_1^{M_1(M,\bar{U}_x)} + 1$ many stages there, a contradiction. This last statement also holds in M_2, since $\mathbb{R} \cap M_2$ is closed under the $M_1^\#$ operator.

The rest of the argument inside M_2 is as in the $n = 1$ case. By 6.8, the resulting definition for the homogeneously Suslin set is in fact correctly-$\boldsymbol{\Delta}^1_3$. This finishes the M_2 case.

Now consider M_3. Things work basically as for M_2; however the definition of Π^1_4-iterability has to differ from that of Π^1_3 because Π^1_4 isn't normed. The reader should consult [23] for the elegant solution. Otherwise the only thing to check is that the $\Delta^1_4(M)$ definition obtained in M_3 extends to one over V. Since M_3 is only Σ^1_4-correct, this isn't as immediate as for M_2. However, defining Π^1_4-iterability requires only $\boldsymbol{\Delta}^1_2$-determinacy, and the resulting closure of Σ^1_3 under $\forall y \in \Delta^1_3(x)$. The statement "$\boldsymbol{\Delta}^1_2$ determinacy holds" is Π^1_4, so its truth in M_3 implies it in V. Investigating the proofs of 4B.3 and 6B.1 of [12], one sees that M_3 and

V therefore agree about the definition of the resulting norm for Π^1_3, and in turn, the Π^1_3 definition of the quantifier $\exists y \in \Delta^1_3(x)$. So the $\Sigma^1_4(M)$ and $\Pi^1_4(M)$ formulae defining the homogeneously Suslin set in M_3 have the same interpretation (in terms of iterability and correct iterates) in V. So the argument that (3) is equivalent to (4) also works in V, as desired. M_4 and beyond involve no new ideas. $\square$(Theorem 6.3)

Finally, for models of ZFC in the region of $0^\P$ or below, we do get an exact characterization of the homogeneously Suslin sets.

Definition 6.13. $0^\P$ is the least active mouse N such that $N|\text{crit}(F^N)$ satisfies "there is a strong cardinal".

Theorem 6.14. *Let $N \models \mathsf{ZFC}$ be an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse satisfying "if $\mu < \kappa$ are measurables, then μ is not strong to κ". Then in N, all homogeneously Suslin sets are $\mathbf{\Pi}^1_1$. In particular, this holds if $N \models \mathsf{ZFC}$ and is below $0^\P$.*

Proof. This is a corollary of the proof of 6.3 and that below $0^\P$, every mouse is an iterate of its core, probably due to Jensen. We give a proof of the latter in our setting. With notation as in the proof of 6.3, consider the comparison of M with $\bar{U}_z$, for some iterable $U = \bar{U}_z$, with last model Q. We claim $U = Q$. Otherwise let $\kappa = \text{crit}(i_{U,Q})$. Then κ is measurable in U, so no $\mu < \kappa$ is strong to κ in U, or therefore in Q. Therefore κ cannot be overlapped by any extender used in $\mathcal{T}$. Now there's $E = E^{\mathcal{T}}_\alpha$ used on $\mathcal{T}$'s main branch with $\text{crit}(E) = \kappa$, since $\kappa \notin \text{Hull}^Q_\omega(\kappa)$. But U, $M^{\mathcal{T}}_\alpha$ and Q agree about $\mathcal{P}(\kappa)$, and $\mathcal{P}(\kappa) \cap Q \subseteq \text{Hull}^Q(\kappa)$, which leads to E being compatible with the first extender used on the U side.

So $\bar{U}_z$ is iterable iff $\bar{U}_z$ is a correct iterate of M, iff it is a wellfounded iterate of M, which is $\Pi^1_1(M)$. $\square$(Theorem 6.14)

7 The Copying Construction & Freely Dropping Iterations

Here we identify and solve some problems with the copying construction of [11]. Actually, since it doesn't take much more work, we prove a generalization of "every initial segment of a mouse is a mouse", since this is really needed for the iterability of the phalanges used in the previous sections.

First we'll briefly discuss the problems ignored in [11], and give examples where these arise.

Suppose M is a type 3 premouse and $\pi : M^{\mathrm{sq}} \to N^{\mathrm{sq}}$ is a lifting map being used during a copying construction. It might be that the exit extender E from M has $\nu_M < \mathrm{lh}(E) < \mathrm{OR}^M$, so $E \notin \mathrm{dom}(\pi)$, but E is not the active extender of M. [11] ignores this. Let $\psi : \mathrm{Ult}(M^{\mathrm{sq}}, F^M) \to \mathrm{Ult}(N^{\mathrm{sq}}, F^N)$ be the canonical embedding. If one has $\psi(\nu_M) \leq \nu_N$, the natural solution is to let $\psi(E)$ be the exit extender from N; otherwise one can first use F^N in the upper tree, then use $\psi(E)$.

Another problem arises if $\psi(\nu_M) < \nu_N$ and F^M is the exit extender from M. Here [11] uses F^N as the exit extender from N. But we have $\psi(\mathrm{lh}(F^M)) < \nu_N$, so the next extender E' used below might be such that $\psi(\mathrm{lh}(E')) < \nu_N$. This causes a break in the increasing length condition of the upper tree.

We now give an example of these two situations. Suppose $\nu_M = [a, f]^M_{F^M}$; then $\psi(\nu_M) = [\pi(a), \pi(f)]^N_{F^N}$. The statement "$[a, f] \geq \nu$" is Π_1; "$[a, f] \leq \nu$" is Π_2. So if π is Π_2-elementary then $\psi(\nu_M) = \nu_N$ but it seems it might be that

(a) π is a 0-embedding and $\psi(\nu_M) > \nu_N$; or

(b) π is a weak 0-embedding and $\psi(\nu_M) < \nu_N$.

Now suppose $\kappa = \mathrm{cof}^M(\nu_M) < \rho_1^M$, κ is measurable in M, and M is 1-sound. Let E be an extender over M with crit κ. Let $i_0 : M^{\mathrm{sq}} \to \mathrm{Ult}_0(M^{\mathrm{sq}}, E)$, $i_1 : M^{\mathrm{sq}} \to \mathrm{Ult}_1(M^{\mathrm{sq}}, E)$, and $\tau : \mathrm{Ult}_0 \to \mathrm{Ult}_1$ be the canonical maps. Then i_0 is cofinal. Let $f : \kappa \to \nu_M \in M$ be cofinal, strictly increasing and continuous. Let $f = [a, g_f]^M_{F^M}$. Then $\nu_M = \sup \mathrm{rg}([a, g_f])$ in $\mathrm{Ult}(M, F^M)$. $[i_0(a), i_0(g_f)]$ represents a strictly increasing, continuous function in $\mathrm{Ult}(\mathrm{Ult}_0, F^{\mathrm{Ult}_0})$, with domain $i_0(\kappa)$; denote this by $i_0(f)$. Now i_0 is cofinal in ν_{Ult_0}, and for any $\alpha < \kappa$, $i_0(f(\alpha)) = i_0(f)(\alpha)$, so $i_0(f)(\kappa) = \nu_{\mathrm{Ult}_0}$. Therefore $\sup \mathrm{rg}(i_0(f)) > \nu_{\mathrm{Ult}_0}$. Since i_0 is a 0-embedding, it is an example of (a).

Now i_1 is a 1-embedding, so is Π_2-elementary, so $\sup \mathrm{rg}(i_1(f)) = \nu_{\mathrm{Ult}_1}$ (where $i_1(f)$ is defined as for $i_0(f)$). Therefore $\tau(\nu_{\mathrm{Ult}_0}) < \nu_{\mathrm{Ult}_1}$. It's easy to check that τ is a weak 0-embedding, so it is an example of (b).

Now we consider another problem with the copying construction. Suppose $\pi : M \to N$ is a weak $k+1$-embedding, and M and N have degree $k+1$ in some iteration trees. If E is applied to M with $\mathrm{crit}(E) = \kappa$, and $\rho^M_{k+1} \leq \kappa$, but $\pi(\kappa) < \rho^M_{k+1}$, then E triggers a drop in degree in the lower tree, but the lifted extender, with crit $\pi(\kappa)$, should not cause a drop if the upper tree is to be normal. So we are forced to let the degrees of corresponding ultrapowers differ between trees; we will see this works fine.

It seems these situations can arise in the proofs of condensation and solidity of the standard parameter (see ([22], §5) and [11]). In the case of condensation, suppose $\sigma : H \to M$ is the (fully elementary) embedding under consideration, with $\mathrm{crit}(\sigma) = \rho = \rho_\omega^H = (\kappa^+)^H$. In the proof, a tree $\mathcal{T}$ on the phalanx (M, H, ρ) is lifted to a tree $\mathcal{U}$ on M, using σ and id as initial lifting maps. $\mathrm{lh}(E_0^\mathcal{T}) > \rho$, so $\mathrm{lh}(E_0^\mathcal{U}) > (\kappa^+)^M$. Say $E = E_\alpha^\mathcal{T}$ has crit κ. Then E measures exactly $\mathcal{P}(\kappa)^H$, but goes back to M, so the tree drops to the least $M|\xi \trianglerighteq M|\rho$ projecting to κ. However, $E_\alpha^\mathcal{U}$ measures all of M, so $\mathcal{U}$ does not experience a drop. We can naturally set $\pi_{\alpha+1} : M_{\alpha+1}^\mathcal{T} \to i_{M,\alpha+1}^\mathcal{U}(M|\xi)$; $\pi_{\alpha+1}$ will be a weak k-embedding, where $\deg^\mathcal{T}(\alpha+1) = k$. Now it might be that $M|\xi$ is type 3, with

$$\rho_{k+1} = \rho_{k+1}^{M|\xi} \leq \kappa < \rho_k^M = \rho_k \ \ \& \ \ \mathrm{cof}^{M|\xi}(\rho_k) = \kappa.$$

Then similarly to the earlier example, $i_{M|\xi,\alpha+1}^\mathcal{T}$ is discontinuous at $\rho_k^{M|\xi}$ and

$$\pi_{\alpha+1}(\rho_k^{M_{\alpha+1}^\mathcal{T}}) < \rho_k^{i_{M,\alpha+1}^\mathcal{U}(M|\xi)}.$$

Suppose the exit extender F from $M_{\alpha+1}^\mathcal{T}$ has

$$\rho_k^{M_{\alpha+1}^\mathcal{T}} \leq \mathrm{crit}(F) < i_{M|\xi,\alpha+1}^\mathcal{T}(\rho_k^{M|\xi}),$$

and measures all of $M_{\alpha+1}^\mathcal{T}$. Then F applies normally to $M_{\alpha+1}^\mathcal{T}$, dropping to degree $k-1$, but

$$\rho_{k+1}^{i_{M,\alpha+1}^\mathcal{U}(M|\xi)} = i_{M,\alpha+1}^\mathcal{U}(\kappa) < \mathrm{crit}(\pi_{\alpha+1}(F)) < \rho_k^{i_{M,\alpha+1}^\mathcal{U}(M|\xi)}.$$

It's also easy to adapt this to give the situation with type 3 premice described earlier.

Before proceeding to give details of a copying construction dealing with the above problems, we show that for simple enough copying none of the above problems occur, so things work exactly as described in [11]. (As seen above though, copying isn't as smooth in general for lifting iterations on phalanges.)

Theorem 7.1. *Suppose $\pi : M \to N$ is a near k-embedding such that if M is type 3, then π preserves representation of ν; i.e. $\psi_\pi(\nu_M) = \nu_N$. Then a normal tree on M lifts to a normal tree on N by the prescription in [11]. Moreover, for each α, π_α is a near $\deg^\mathcal{T}(\alpha)$-embedding, and if $M_\alpha^\mathcal{T}$ is type 3, π_α preserves representation of ν_{M_α}.*

Proof. As in [15], the nearness of embeddings is maintained inductively. This immediately knocks out the problems of differing degrees and possibility (b) above. (Note "$\alpha < \rho_k$" is Σ_{k+1}, so if $\pi : M \to N$ is Σ_{k+1}-elementary then $\pi(\rho_k^M) \geq \rho_k^N$.) Since "$\nu$-preservation" is preserved when taking ultrapowers of degree ≥ 1, we consider only degree 0.

Suppose M is type 3, $\kappa < \nu_M$, and E is an extender over M with $\mathrm{crit}(E) = \kappa$. Say $\pi : M^{\mathrm{sq}} \to N^{\mathrm{sq}}$ is our copy map, and E is lifted to F over N. Let $\tau : \mathrm{Ult}_0(M^{\mathrm{sq}}, E) \to \mathrm{Ult}_0(N^{\mathrm{sq}}, F)$ be as usual. We need that τ preserves ν-representation.

In the case that ν_M is singular in M, this is straightforward to show using the approach of the earlier example (which lead to examples of (a) and (b) above); we leave it to the reader.

Suppose ν_M is regular in M. Here we simply show that i_E preserves ν-representation; this suffices. Fixing $h : \mu^M \to \mu^M \in M$ and $\gamma \in M$, notice there are only boundedly many $\alpha < \nu_M$ of the form $i_{F^M}(h)(c)$ for $c \in [\gamma]^{<\omega}$. Let $\delta_{h,\gamma}$ be the supremum of these ordinals. The same applies to a μ^M-sequence of functions in M. Let $[a, f] = \nu_M$, $U^{\mathrm{sq}} = \mathrm{Ult}_0(M^{\mathrm{sq}}, E)$ and $b, g \in U^{\mathrm{sq}}$; suppose $[b, g]^U_{F^U} < [i(a), i(f)]^U_{F^U}$. Let $\gamma \in M$ be such that $b \in [i_E(\gamma)]^{<\omega}$ and $a \subseteq \gamma$.

If $\mu^M < \kappa$, then we may assume $i_E(g) = g \in M$. Let $\delta = \delta_{g,\gamma}$. Let $F' = F \restriction \delta$. Then F' encodes the bounding property of δ:

$$\forall u \in [\gamma]^{<\omega} \ ([u, g] < [a, f] \implies [u, g] < \delta).$$

Clearly this is preserved by i_E, so $[b, g] < i_E(\delta) < \nu_U$.

If $\kappa \leq \mu^M$, let $g = i_E(G)(u)$, where $G \in M$ produces only functions $\mu^M \to \mu^M$. Then using a bound $\delta_{G,\gamma}$, the same argument works. □(Theorem 7.1)

Definition 7.2. Let the *free (earliest model)* (n, θ, λ) iteration game be as follows. The players build a stack of λ iteration trees, one in each round, with II choosing wellfounded branches and creating wellfounded limits at all limit stages. If in round α, I is to play, and $\mathcal{T}$ is the current tree, with models M_γ and exit extenders E_γ, $\mathcal{T}$ with length $\delta + 1$, he

- May move to the next round (and must if the present tree has length θ); in this case he also chooses an initial segment of M_δ and a degree $n \leq \omega$ for the root of the next tree, where $n \leq \deg(\delta)$ if M_δ is chosen;
- Must choose an extender E from the present model with length greater than all those already played in the current round;
- Must choose some $\beta \leq \alpha$, such that $\mathrm{crit}(E) < \nu^{\mathcal{T}}_\beta$ (see the remark below for a clarification); for earliest model iterations, he must choose the least such β;
- Must choose some initial segment $M^* = (M^*)_{\delta+1}$ of M_β with $\mathrm{crit}(E) < \mathrm{OR}^{M^*}$ and $\mathcal{P}(\mathrm{crit}(E)) \cap M^*$ measured by E;
- Must choose $\deg(\delta + 1) = n \leq \omega$ for the ultrapower, such that $\mathrm{crit}(E) < \rho^{M^*}_n$ (clarification below) and $(M^* = M_\beta \implies n \leq \deg(\beta))$.

Payoff is as expected.

Remark 7.3. We allow M^* to be type 3, with $\mathrm{crit}(E) = \nu_{M^*}$. In this case $\deg(\delta + 1) = 0$, and $\mathrm{Ult}_0(M^*, E)$ is formed without squashing M^*, as in §4, Submeasures. If $\delta + 1 \leq_{\mathcal{T}} \gamma$ and $i^{\mathcal{T}}_{\delta+1,\gamma}$ exists, then γ is called *anomalous*. If $E^{\mathcal{T}}_\gamma$ is the active extender of $M^{\mathcal{T}}_\gamma$, we set $\nu^{\mathcal{T}}_\gamma = i^{\mathcal{T}}_{M^*,\gamma}(\mathrm{crit}(E))$, the largest cardinal of $M^{\mathcal{T}}_\gamma$. $\nu = \nu_{E^{\mathcal{T}}_\gamma}$ may be less than $\nu^{\mathcal{T}}_\gamma$ here; in fact $\nu = \sup_{\alpha<\gamma} \nu_{E^{\mathcal{T}}_\alpha}$. $M^{\mathcal{T}}_\gamma$ isn't a premouse: if $\gamma > \delta + 1$ then $M^{\mathcal{T}}_\gamma$ doesn't satisfy the initial segment condition; if $\gamma = \delta + 1$ then ν is less than $M^{\mathcal{T}}_\gamma$'s largest cardinal. But the iteration tree still makes sense. We'll also call $M^{\mathcal{T}}_\gamma$ an *anomalous* structure, and let $\nu_{M^{\mathcal{T}}_\gamma}$ denote ν. When dealing with definability over an anomalous structure, there is no squashing.

Definition 7.4. The maximal (earliest model) iteration game is the game with rules as above, except that player I must always choose $(M^*)_{\delta+1}$, then $\deg(\delta+1)$, as large as possible.

Definition 7.5. The definition of weak m-embedding is given on page 52 of [11], except that the condition "$\rho_m \in X$" shouldn't be there. Here we will also call a $\pi : M \to N$ an *(anomalous) weak* 0*-embedding* when M is an anomalous structure, N is a type 3 premouse, π is Σ_0-elementary, there's a cofinal subset of OR^M on which π is Σ_1-elementary, and $\pi(\nu_M) = \nu_N$. (As stated above, we're not squashing, and ν_M is the largest cardinal of M.)

Theorem 7.6. *Suppose N is maximally (earliest model) (n, θ, λ)-iterable, and that $\pi : M \to N|\eta$ is a weak m-embedding, where $m \leq n$ if $\eta = \mathrm{OR}^N$. Then M is freely (earliest model) (m, θ, λ)-iterable.*

Remark 7.7. We leave to the reader the second half of the copying we require for the applications of earlier sections. That is the copying of an iteration on some phalanx to a freely-dropping one on N. It's not much different to the construction of 7.6.

Proof of Theorem 7.6. The idea is just to reduce a free iteration on M to a normal one on N via a copying construction. There are more details than usual - but mostly due to patches for the usual copying construction.

Given a (possibly anomalous) weak 0-embedding $\pi : M \to N$ between active premice (or their squashes in the non-anomalous type 3 case), let

$$\psi_\pi : \mathrm{Ult}(M|((\mu^M)^+)^M, F^M) \to \mathrm{Ult}(N|((\mu^N)^+)^N, F^N)$$

be the canonical embedding. Note $\pi \subseteq \psi_\pi$, and when M is type 1 or 2, $\psi_\pi(\mathrm{lh}(F^M)) = \mathrm{lh}(F^N)$.

Let's assume that $\lambda = 1$. As $\mathcal{T}$ is being built on M, with models M_α and extenders E_α, we build $\mathcal{U}$ on $N_0 = N$, with models N_α and extenders F_α. One new detail is that $\mathcal{U}$ may have nodes in its tree order not corresponding to extenders used in $\mathcal{T}$. For bookkeeping purposes, $\mathcal{T}$ and $\mathcal{U}$ will in general be padded. If $E_\beta = \emptyset$, we'll have M_β is type 3, and we'll set

$$M_{\beta+1} = M_\beta||\mathrm{OR}^{M_\beta},$$

the reduct of M_β. We'll also set $\nu^{\mathcal{T}}_\beta = \nu_{M_\beta}$ and have $\nu^{\mathcal{T}}_\beta \leq \nu^{\mathcal{T}}_{\beta+1} < \mathrm{lh}(E_{\beta+1})$. So an extender with crit in $[\nu_{M_\beta}, \nu^{\mathcal{T}}_{\beta+1})$ applies to a proper segment of $M_{\beta+1}$, since $M_{\beta+1}$ is passive, and $\nu_{M_\beta+1}$ is its largest cardinal. Thus we will never take an ultrapower of the entire $M_{\beta+1}$. (So $\deg^{\mathcal{T}}(\beta+1)$ isn't relevant; we can set $\mathcal{T}-\mathrm{pred}(\beta+1) = \beta$.)

For any α, there'll be an ordinal η_α and a (possibly anomalous) weak $\deg^{\mathcal{T}}(\alpha)$-embedding

$$\pi_\alpha : M_\alpha \to N_\alpha|\eta_\alpha.$$

If M_α is anomalous, we'll have that $\eta_\alpha < \mathrm{OR}^{N_\alpha}$. If M_α is active let $\psi_\alpha = \psi_{\pi_\alpha}$; otherwise let $\psi_\alpha = \pi_\alpha$. When $E^{\mathcal{T}}_\alpha \neq \emptyset$, we'll have a structure R_α and a Σ_0-elementary

$$\psi_\alpha \restriction M_\alpha|\mathrm{lh}(E_\alpha) : M_\alpha|\mathrm{lh}(E_\alpha) \to R_\alpha.$$

Here when E_α is type 3, the map is to literally apply to $M_\alpha|\mathrm{lh}(E_\alpha)$, not its squash. Usually $R_\alpha = N_\alpha|\mathrm{lh}(F_\alpha)$. In any case, ψ_α provides a map lifting E_α to (possibly a trivial extension of) F_α, sufficient for the proof of the shift lemma.

For $\beta < \alpha$, we'll maintain:

$$\psi_\beta \restriction (\mathrm{lh}(E_\beta)+1) \subseteq \pi_\alpha. \tag{12}$$

$$\psi_\beta(\nu^{\mathcal{T}}_\beta) \geq \nu^{\mathcal{U}}_\beta. \tag{13}$$

$$\psi_\beta``\nu^{\mathcal{T}}_\beta \subseteq \nu^{\mathcal{U}}_\beta. \tag{14}$$

And when $E_\beta \neq \emptyset$,

$$\psi_\beta(\mathrm{lh}(E_\beta)) \geq \mathrm{lh}(F_\beta). \tag{15}$$

We start at $\pi_0 = \pi$. Suppose we have everything up to stage α. Say player I chooses an exit extender E from M_α. Let $E_\alpha = E$ except for in the last case below, in which $E_\alpha = \emptyset$. We need to define F_α (and in the last case, $F_{\alpha+1}$). Let $M = M_\alpha$, $N = N_\alpha|\eta_\alpha$, $\pi = \pi_\alpha$ and $\psi = \psi_\alpha$.

Case 1. $E \in \mathrm{dom}(\pi)$ or E is the active type 1 or 2 extender of M.

Here F_α is defined as usual: If $\mathrm{lh}(E) \in \mathrm{dom}(\pi)$, let $F_\alpha = \pi(E)$. Otherwise let F_α be the active extender of N. By the preservation hypotheses (13), we then have $\mathrm{lh}(F_\alpha)$ is larger than previous extenders on $\mathcal{U}$. Also $R_\alpha = N|\mathrm{lh}(F_\alpha)$.

Case 2. E is the active type 3 extender of M and $\psi(\nu_M) \geq \nu_N$, or E is the active extender of an anomalous M.

Let $F_\alpha = F^N$ and

$$R_\alpha = (N|(\psi(\nu_M)^+)^{\mathrm{Ult}(N,F^N)}, (F^N)^*),$$

where $(F^N)^*$ is the extender of length $(\psi(\nu_M)^+)^{\mathrm{Ult}(N,F^N)}$ derived from i_{F^N}, coded amenably. If $\psi(\nu_M) = \nu_N$, R_α is just N. Otherwise R_α isn't a premouse, but we still get

$$\psi \restriction M : M \to R_\alpha$$

is Σ_0-elementary. This is sufficient for the proof of the shift lemma. (This may seem unnecessary, since we already had all the generators of E within the domain of π. But dropping to a level below $((\mu^M)^+)^M$, and applying $E \restriction \nu_E$, yields a smaller ultrapower than applying E; in particular, the smaller ultrapower does not agree with M below $\mathrm{lh}(E)$.) Note that when M is anomalous, our assumptions on π give $\psi(\nu^{\mathcal{T}}_\alpha) = \nu^{\mathcal{U}}_\alpha$.

Case 3. M is a premouse, E is its active type 3 extender, and $\psi(\nu_M) < \nu_N$.

Here we can't set $F_\alpha = F^N$, as discussed earlier. We use a proper segment of F^N instead. $F^N \restriction \sup \pi``\nu_M$ may not be on N's sequence, but it is reasonable to set

$$F_\alpha = \mathrm{tc}(F^N \restriction \psi(\nu_M)).$$

This segment is in fact on N's sequence, since $\psi(\nu_M)$ is a cardinal of N. Clearly ψ factors through

$$\psi' : \mathrm{Ult}(M|(\mu^M)^+, F^M) \to \mathrm{Ult}(N|(\mu^N)^+, F_\alpha)$$

and the canonical map $\mathrm{Ult}(N, F_\alpha) \to \mathrm{Ult}(N, F^N)$ has crit $\mathrm{lh}(F_\alpha)$. It follows that setting $R_\alpha = N|\mathrm{lh}(F_\alpha)$ works.

In either of the previous cases, if $\beta < \alpha$, then $\mathrm{lh}(E_\beta) < \mathrm{OR}^{M^{\mathrm{sq}}} = \nu_M$, so the agreement hypotheses give $\mathrm{lh}(F_\beta) < \sup \pi``\nu_M < \mathrm{lh}(F_\alpha)$, maintaining the increasing length condition of $\mathcal{U}$.

Case 4. M is a type 3 premouse, $\mathrm{OR}^{M^{\mathrm{sq}}} < \mathrm{lh}(E) < \mathrm{OR}^M$ and $\psi_\pi(\nu_M) \leq \nu_N$.

Here $\psi_\pi(\mathrm{OR}^M) \leq \mathrm{OR}^N$. Let $E_\alpha = F_\alpha = \emptyset$, $\nu_\alpha^{\mathcal{T}} = \nu_M$ and $\nu_\alpha^{\mathcal{U}} = \psi(\nu_M)$. Let

$$M_{\alpha+1} = M||\mathrm{OR}^M,$$

and

$$N_{\alpha+1} = N||\psi(\mathrm{OR}^M) = N||(\psi(\nu_M)^+)^N.$$

Let $\pi_{\alpha+1} = \psi \restriction \mathrm{OR}^M$. Now set $F_{\alpha+1} = \psi_\pi(E)$. Again the increasing length condition follows from (13) and we set $R_\alpha = N|\mathrm{lh}(F_\alpha)$.

Case 5. M is type 3, $\mathrm{OR}^{M^{\mathrm{sq}}} < \mathrm{lh}(E) < \mathrm{OR}^M$ and $\psi_\pi(\nu_M) > \nu_N$.

Here it is not clear that any extender on N's sequence corresponds to E. A solution is to use an extra ultrapower in the upstairs tree. Set $E_\alpha = \emptyset$ but $F_\alpha = F^N$. Let $N_{\alpha+1}$ be the maximal degree ultrapower of the model as chosen in a normal tree. Let $\eta_{\alpha+1} = \psi(\mathrm{OR}^M)$ and $\pi_{\alpha+1} = \psi \restriction \mathrm{OR}^M$. (Note $N_{\alpha+1}$ agrees with $\mathrm{Ult}(N, F^N)$ past $\eta_{\alpha+1}$.) Note $\psi(\nu_M)$ is the largest cardinal of $N_{\alpha+1}|\eta_{\alpha+1}$. Now set $F_{\alpha+1} = \pi_{\alpha+1}(E)$. Increasing length holds as

$$\pi_{\alpha+1}(\mathrm{lh}(E)) > \pi_{\alpha+1}(\nu_M) \geq \mathrm{lh}(F^N)$$

by case hypothesis, and that $\pi_{\alpha+1}(\nu_M)$ is a cardinal of $\mathrm{Ult}(N, F^N)$. Here $R_{\alpha+1} = N_{\alpha+1}|\mathrm{lh}(F_{\alpha+1})$.

This defines F_α in all cases, and $F_{\alpha+1}$ where needed. We now notationally assume there was no padding used (so $F_{\alpha+1}$ is not yet defined), but otherwise the same discussion holds with $\alpha + 1$ replacing α.

Suppose player I chooses appropriate $M^* = (M^*)_{\alpha+1} = M_\beta|\xi$ and $\deg(\alpha + 1) = n$. Let $P \trianglelefteq M_\beta$ be the longest possible that E_α can apply to, so $(M^*)_{\alpha+1} \trianglelefteq P$. Let $\kappa = \mathrm{crit}(E_\alpha) < \nu_\beta^{\mathcal{T}}$. Our use of padding gives $\kappa \in \mathrm{dom}(\pi_\beta)$. Let

$$\kappa' = \mathrm{crit}(F_\alpha) = \pi_\alpha(\kappa) = \pi_\beta(\kappa) < \nu_\beta^{\mathcal{U}}.$$

(The inequality follows the agreement hypotheses.)

If we're dealing with earliest model trees and $\gamma < \beta$, then $\nu_\gamma^{\mathcal{T}} \leq \kappa$. So preservation gives $\nu_\gamma^{\mathcal{U}} \leq \kappa'$, and N_β is the correct model to return to in $\mathcal{U}$.

Let $N^* = (N^*)_{\alpha+1} = P' \trianglelefteq N_\beta$ be largest measured by F_α. If $M^* = P = M_\beta$ let $\eta_{\alpha+1}^* = \eta_\beta$; otherwise let $\eta_{\alpha+1}^* = \psi_\beta(\mathrm{OR}^{M^*})$. We need to check $\eta_{\alpha+1}^* \leq \mathrm{OR}^{P'}$, so that M^* is embedded into a level Q^* of N^*.

If $E_\beta = \emptyset$ then $P = M_\beta$ as $\nu_\beta^{\mathcal{T}} = \nu_{M_\beta}$ is a cardinal of M_β. Also $N_\beta|\eta_\beta \trianglelefteq P'$ (even when $\psi_\beta(\nu_\beta^{\mathcal{T}}) < \nu_{N_\beta|\eta_\beta}$, $\psi_\beta(\nu_\beta^{\mathcal{T}})$ is still a cardinal of $N_\beta|\eta_\beta$).

So assume $E_\beta \neq \emptyset$. Suppose $\mathrm{lh}(E_\beta) \in \mathrm{dom}(\pi_\beta)$. If $P \lhd M_\beta$ then $P \in \mathrm{dom}(\pi_\beta)$, and P is the least P_1 such that $M_\beta|\mathrm{lh}(E_\beta) \trianglelefteq P_1 \lhd M_\beta$ and P_1 projects to κ. It follows that $\pi_\beta(P) = P'$. If instead $P = M_\beta$ then $(\kappa^+)^{M_\beta} < \mathrm{lh}(E_\beta)$ and $\pi_\beta((\kappa^+)^{M_\beta}) = (\kappa'^+)^{N_\beta|\eta_\beta}$, so $N_\beta|\eta_\beta \trianglelefteq P'$.

Now suppose E_β is the active extender of M_β. So $P = M_\beta$. If F_β isn't active on $N_\beta|\eta_\beta$, then case 3 applies, and the cardinality of $\psi_\beta(\nu^{\mathcal{T}}_\beta)$ in $N_\beta|\eta_\beta$ gives $N_\beta|\eta_\beta \trianglelefteq P'$.

This covers all cases. Let $Q^* = N^*|\eta^*_{\alpha+1}$. If M^* is anomalous, we inductively have $Q^* \lhd N_\beta$. Note $\kappa < \nu_{M^*}$, so $\kappa' < \nu_{Q^*}$. Since ν_{Q^*} is the image of a critical point leading to N_β, it's a cardinal of N_β. Therefore $Q^* \lhd N_\beta = (N^*)_{\alpha+1}$. If M^* isn't anomalous but $\alpha+1$ will be, then M^* is below $(\kappa^+)^{E_\alpha}$, which gives $Q^* \lhd (N^*)_{\alpha+1}$. This will give $\eta_{\alpha+1} < \mathrm{OR}^{N_{\alpha+1}}$ later.

Letting $\pi^* = \pi^*_{\alpha+1} = \psi_\beta \restriction M^*$,

$$\pi^* : M^* \to Q^*$$

is a (possibly anomalous) weak n-embedding ($n = \deg^{\mathcal{T}}(\alpha+1)$). By the agreement hypotheses, ψ_α agrees with π^* on $\mathcal{P}(\kappa)$. So the shift lemma goes through with π^* and $\psi_\alpha : M_\alpha|\mathrm{lh}(E_\alpha) \to R_\alpha$. So there's a weak n-embedding

$$\tau : \mathrm{Ult}_n(M^*, E) \to \mathrm{Ult}_n(Q^*, F)$$

such that $\tau \circ i^{M^*}_{E,n} = i^{Q^*}_{F,n} \circ \pi^*$. (If M^* is anomalous, both ultrapowers are to be formed at the unsquashed level. It doesn't really matter here whether $i^{Q^*}_{F,n}(\nu_{Q^*})$ is the sup of generators of $\mathrm{Ult}_0(Q^*, F)$, but it is, since ν_{Q^*} is regular in Q^*.)

Agreement.

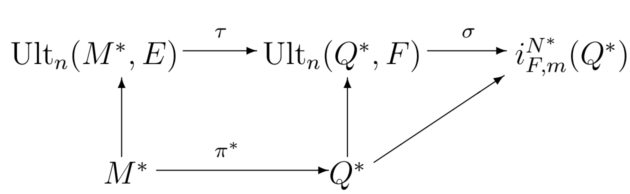
We have (13), (14) and (15) hold at α by definition of F_α. $\psi_\alpha \restriction \mathrm{lh}(E_\alpha) + 1 \subseteq \tau$ by definition of τ. We will now define $\pi_{\alpha+1} = \sigma \circ \tau$ where $\mathrm{crit}(\sigma) > \mathrm{lh}(F_\alpha)$, which will establish (12) between π_α and $\pi_{\alpha+1}$.

We just have to set

$$\sigma : \mathrm{Ult}_n(Q^*, F) \to i^{N^*}_{F,m}(Q^*);$$
$$\sigma([a,f]^{Q^*}_{F,n}) = [a,f]^{N^*}_{F,m},$$

where $m = \deg^{\mathcal{U}}(\alpha+1)$.

First note that given $[a,f]$ in the domain, we do have that it represents an element of the larger ultrapower. (If $Q^* \lhd N^*$ then $f \in N^*$. If $Q^* = N^*$ then $n \leq m$ since π^* is a weak n-embedding and $\mathrm{crit}(E) < \rho^{M^*}_n$.) Also it's clear that $[a,f]^{N^*}_{F,m}$ is an element of $i^{N^*}_{F,m}(Q^*)$, and that the map is well defined. We get $\mathrm{lh}(F) < \mathrm{crit}(\sigma)$, and the following diagram commutes (with the canonical embeddings):



Now $\sigma \circ \tau$ is a weak m-embedding. If $Q^* = N^*$ and $n = m$, then $\sigma = \text{id}$. If $n = \omega$ then all maps in the diagram are fully elementary. So suppose $n < \omega$. σ is Σ_n elementary by Los' theorem. The preservation properties required of a weak n-embedding hold because we know they hold of all embeddings other than σ in the commuting diagram, and that $i^{M^*}_{E,n}$ is cofinal in the ρ_n of $\text{Ult}_n(M^*, E)$. Given a cofinal $X_1 \subseteq \rho_n^{M^*}$ on which π^* is Σ_{n+1} elementary, $X = i^{M^*}_{E,m}$"X_1 works for $\sigma \circ \tau$. For τ"$X = i^{Q^*}_{F,n}$"$(\pi_\beta$"$X_1)$, and both $i^{Q^*}_{F,n}$ and $i^{N^*}_{F,m} \restriction Q^*$ are Σ_{n+1} elementary, so by commutativity, σ is Σ_{n+1} elementary on τ"X.

This finishes the successor stage of construction. Limit stages are as usual. $\Box$(Theorem 7.6)

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