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HOMOGENEOUSLY SUSLIN SETS IN TAME MICE

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Abstract. This paper studies homogeneously Suslin (hom) sets of reals in tame mice. The following results are established: In $0^\P$ the hom sets are precisely the $\utilde{\Pi}^1_1$ sets. In M_n every hom set is correctly $\utilde{\Delta}^1_{n+1}$, and $(\delta+1)$-universally Baire where δ is the least Woodin. In M_ω every hom set is $<\lambda$-hom, where λ is the supremum of the Woodins.

§1. Introduction. In certain mice, the class of homogeneously Suslin (hom) sets of reals admits analysis not available in V. The first example of this was based on Kunen's analysis of measures in $L[U]$, and is due to Steel:

$$L[U] \models \text{The hom sets are precisely the } \utilde{\Pi}^1_1 \text{ sets.} \tag{1}$$

Here we generalize (1) to various tame mice. Some of our results are exact characterizations and some are partial. We will prove the following.

THEOREM (6.1). *Let N be a mouse modelling ZFC + "For all measurables $\mu < \kappa$, μ is not strong to κ". Then in N, every hom set is $\utilde{\Pi}^1_1$.*

Therefore, $0^\P$ (the sharp for a strong cardinal, see §6) satisfies "the hom sets are precisely the $\utilde{\Pi}^1_1$ sets". Recall that M_n is the canonical inner model for $n<\omega$ Woodin cardinals. Let $\delta_0^{M_n}$ be its least Woodin.

1.1. DEFINITION. (a) Let N be an inner model of ZFC which is Σ^1_{n+1}-correct (i.e., $\mathbb{R}^N \preccurlyeq_{\Sigma^1_{n+1}} \mathbb{R}$). Let $Z \in \mathcal{P}(\mathbb{R})^N$ and $z \in \mathbb{R}^N$. Then Z is *N-correctly-$\Delta^1_{n+1}(z)$* iff there is a $\Delta^1_{n+1}(z)$ set Z' such that $Z = Z' \cap N$.
(b) Let $z \in N = M_n$. The M_n-class of *correctly* $\Delta^1_{n+1}(z)$ sets is the V-class of M_n-correctly-$\Delta^1_{n+1}(z)$ sets.
The associated boldface notions allow any parameter $z \in N$.

In 1.1, the collection of M_n-correctly-$\Delta^1_{n+1}(z)$ sets is defined in V. Part (b) makes sense because by 2.2, it is a class of M_n, defined uniformly from only z.

THEOREM (3.2). *In M_n, every hom set of reals is correctly $\utilde{\Delta}^1_{n+1}$.*

COROLLARY (3.3; Steel, Sargsyan, S.). *In M_n, every hom set of reals is (δ_0+1)-universally Baire.*

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In M_n, all $\utilde{\Pi}^1_n$ sets are hom, by [8], but an exact descriptive characterization of the hom sets there remains elusive. However:

Corollary (3.4). *In M_n, a set of reals is weakly hom iff it is $\utilde{\Sigma}^1_{n+1}$.*

Looking higher, Steel and Woodin observed that our argument adapts to M_ω. The author then proved a generalization for various tame mice. [1] In such cases we obtain a precise characterization. For example:

Theorem (6.5; Steel, Woodin, S.). *Let λ be the sup of the Woodins in M_ω. Then in M_ω, the hom sets are precisely the $<\lambda$-hom sets, which are precisely the correctly $(\utilde{\Delta}^2_1)^{L(\mathbb{R})}$ sets (see 6.4).*

Let N be the least non-tame mouse, and $\kappa = \mathrm{crit}(F^N)$. Then in N, the hom sets are precisely the $<\kappa$-hom sets.

Finally, we do have a partial result outside the realm of tame mice:

Theorem (6.7). *In M_{wlim}, every δ_0-hom set is $<\lambda$-hom, where λ is the sup of the Woodins.*

Other related results have been known for some time (we won't need these, however). Woodin proved the following version of 3.4:

Fact (Woodin, [2]). (AD+DC) *Suppose μ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. For measure one many $\sigma \in \mu$, if g is a generic enumeration of $\mathbb{R} \cap M_1(\sigma)$ in order-type $\omega_1^{M_1(\sigma)}$, then "weakly hom" coincides with $\utilde{\Sigma}^1_2$ in $M_1(\sigma)[g]$.*

The analogous statement about $M_n(\sigma)$ and $\utilde{\Sigma}^1_{n+1}$ sets also holds.

Lower in the mouse order, Schindler and Koepke generalized (1) in two ways:

Fact (Schindler, Koepke, [5]). *Suppose either (a) [$\neg\exists 0^{\mathrm{long}}$];[2] or (b) [$\neg\exists 0^{¶}$ and $V = K$ and K is below a μ-measurable]. Then every hom set is $\utilde{\Pi}^1_1$.*

In fact they prove the same with a more general class than "hom", and note that (a) does not require that V be a premouse.

The paper proceeds as follows. In §2 we discuss some tools we'll need: correctly $\utilde{\Delta}^1_{n+1}$ sets, finite support for iteration trees, and some fine structure. In §3 we give a key lemma that allows us, within the mice we're considering, to reduce the wellfoundedness of towers of measures to the iterability of countable premice (or something close to this). We'll then apply this to the study of the hom sets in M_n, proving the first clause of 3.2. In §4 we complete the proof of 3.2 by showing, in M_n, the equivalence of "correctly $\utilde{\Delta}^1_{n+1}$" with "$(\delta_0^{M_n}+1)$-universally Baire". In §5 we extend the results of §3 a little for M_1. In §6 we adapt the arguments to the cases of $0^{¶}$, M_ω and some higher tame mice.

1.1. Background, conventions and notation. Moschovakis' book [10] covers the descriptive set theory we use.

We assume basic familiarity with the notions of *homogeneously Suslin* and *universally Baire* sets. Good introductions are provided either by [19, §§1,2] or by [6, §§1.1,1.2,1.3 and the first part of §3.3].

[1] Thanks to John Steel for the suggestion to look into this.

[2] That is, 0^{long} does not exist. 0^{long} and a μ-measurable are both below $0^{¶}$. See [5].

A *tower* (*of measures*) μ *on* P is a pair $(\langle\mu_n\rangle_{n<\omega}, \langle i^\mu_{m,n}\rangle_{m\leq n<\omega})$ such that $\mu_n \in P$, P satisfies "μ_n is a countably complete ultrafilter", and the maps $i^\mu_{m,n} : \mathrm{Ult}(P, \mu_m) \to \mathrm{Ult}(P, \mu_n)$ form a commuting system of embeddings.

We often use *hom* to abbreviate *homogeneously Suslin.*

Most inner model theory we assume is covered in [20]; familiarity with parts of [15], and isolated elements of [14], [16], [12] and [18], is assumed at some points. We use the definition of *premouse* given in [20]; in particular, all premice are fine structural. Let P be a premouse. Then $\mathbb{E}^P$ denotes the extender sequence of P and F^P denotes the active extender. For $\alpha \leq \mathrm{OR}^P$, $P|\alpha$ denotes the initial segment of P with ordinal height α; $P||\alpha$ is the passive version.

In discussing definability over P, our use of "Σ_n" and "Π_n" is really abbreviating "$r\Sigma_n$" and "$r\Pi_n$". The function given by the nested Skolem term τ (i.e. composition of a sequence of Skolem terms) and parameter q is denoted $f_{\tau,q}$. Assuming P is $(n-1)$-sound (or $n = \omega$-sound), and given $X \subseteq P$, $\mathrm{Def}^P_n(X)$ is the set of points in P of the form $f_{\tau,\emptyset}(a)$ for some n-term (or term) τ and $a \in X^{<\omega}$. $\mathrm{Hull}^P_n(X)$ is the transitive collapse of $\mathrm{Def}^P_n(X)$. An ω-(*pre*)*mouse* is a (pre)mouse which is ω-sound and projects to ω.

Given an iteration tree $\mathcal{T}$ on P with a last model, $i^{\mathcal{T}}$ is the main branch embedding if this branch does not drop in model, and is undefined otherwise. A branch b thru $\mathcal{T}$ is $\mathcal{T}$-*maximal* iff $\forall\lambda < \mathrm{lh}(\mathcal{T})[b \neq \{\alpha < \mathrm{lh}(\mathcal{T}) \mid \alpha <_{\mathcal{T}} \lambda\}]$. $\mathcal{T}$ is *above* δ if all the extenders of $\mathcal{T}$ have critical points $\geq \delta$.

If context determines a particular embedding $P \to Q$, this is denoted $i_{P,Q}$.

Given premice P, Q, a pair of trees $(\mathcal{T},\mathcal{U})$ is a *partial comparison* of P vs Q iff $\mathcal{T}$ is on P, $\mathcal{U}$ is on Q, and the trees constitute an initial segment of a comparison of P and Q. We similarly define (*successful*) *comparison.* Given strategies Σ, Γ, a (partial) comparison $(\mathcal{T},\mathcal{U})$ is *via* (Σ, Γ) iff $\mathcal{T}$ is via Σ and $\mathcal{U}$ is via Γ.

If P has exactly $n < \omega$ Woodin cardinals, these are denoted $\delta^P_0 < \ldots < \delta^P_{n-1}$. If $n = 1$, $\delta^P = \delta^P_0$. Likewise when P has ω Woodins. When we write "M_n" we implicitly assume that $n < \omega$; we only refer to M_ω explicitly, by "M_ω".

§2. Preliminaries.

2.1. Correctly $\underset{\sim}{\Delta}^1_{n+1}$ sets in M_n. The following is a correctness result for M_n.[3] If δ is Woodin and $\kappa < \delta$, let $\mathbb{B}^\delta_\kappa$ denote the extender algebra at δ using critical points $\geq \kappa$, and x^δ_κ a name for the generic real. Let $\mathbb{B}^\delta = \mathbb{B}^\delta_0$ and $x^\delta = x^\delta_0$.

2.1. Fact (Woodin). *Let $n \in \omega$, N be an active mouse with n Woodins $> \alpha$, and $\mathbb{P} \in N|\alpha$. Let G be $\mathbb{P}$-generic over N. Then (a) $N[G]$ is Σ^1_{n+1}-correct, and if n is even, $N[G]$ is Σ^1_{n+2}-correct. Moreover, (b) let $\varphi(x, y, z)$ be Σ^1_n, $b \in \mathbb{R}^{N[G]}$, and $\alpha < \kappa < \delta_0 < \ldots < \delta_{n-1}$ with δ_i Woodin in N. If n is odd,*

$$\exists y\forall z\varphi(b, y, z) \iff N[G] \models \exists p \in \mathbb{B}^{\delta_0}_\kappa \Big[p \Vdash_{\mathbb{B}^{\delta_0}_\kappa} \forall z\varphi(b, x^{\delta_0}_\kappa, z)\Big],$$

and if n is even

$$\exists y\forall z\varphi(b, y, z) \iff N[G] \models \exists p \in \mathbb{B}^{\delta_0}_\kappa \Big[p \Vdash_{\mathbb{B}^{\delta_0}_\kappa} [\emptyset \Vdash_{\mathbb{B}^{\delta_1}_{\delta_0}} \varphi(b, x^{\delta_0}_\kappa, x^{\delta_1}_{\delta_0})]\Big].$$

[3]Thanks to Grigor Sargsyan for pointing out 2.1 to the author, and its relevance to 4.1.

See [15, 4.6] and the argument for [15, 4.10] for the proof. See also [20, 7.14, 7.15, and remarks between] for the main ideas. Another proof of (a) essentially follows the argument for 4.1. [4]

2.2. Remark. We now verify that Definition 1.1(b) makes sense. First, M_n is Σ^1_{n+1}-correct, by 2.1. Now, say a pair of reals (a,b) is a $\undertilde{\Delta}^1_{n+1}$-*code* if a,b code complementary $\undertilde{\Sigma}^1_{n+1}$ sets, in a standard coding of $\undertilde{\Sigma}^1_{n+1}$. Then the statement "$(x,y)$ is a $\undertilde{\Delta}^1_{n+1}$-code" is Π^1_{n+2}. By 2.1, M_n can compute the truth of such statements, so the class of M_n-correctly-$\undertilde{\Delta}^1_{n+1}(z)$ sets is definable over M_n (uniformly from only z), so 1.1(b) does indeed define a class of M_n.

The remarks above and 2.1 show that for n even, in M_n, a set is $\undertilde{\Delta}^1_{n+1}$ iff it is correctly $\undertilde{\Delta}^1_{n+1}$, and in fact, every $\undertilde{\Delta}^1_{n+1}$-code is a "correct" one. However in M_1 there are correctly $\undertilde{\Delta}^1_2$ sets (e.g. $\emptyset$) for which not every $\undertilde{\Delta}^1_2$-code is "correct". This is because M_1 is not Σ^1_3-correct. But the contrast is stronger than this:

2.3. Fact (Folklore). *In M_{2n+1}, not all $\undertilde{\Delta}^1_{2n+2}$ sets are correctly $\undertilde{\Delta}^1_{2n+2}$.*

Proof. Deny. Work in $N = M_{2n+1}$. By 4.1, all $\undertilde{\Delta}^1_{2n+2}$ sets are (δ_0^N+1)-universally Baire. So by Neeman [11, 6.17] they're determined, and by Martin [4, 30.10], $\undertilde{\Pi}^1_{2n+2}$ determinacy follows. So all $\undertilde{\Sigma}^1_{2n+3}$ sets have the Baire property, contradicting the existence of a $\undertilde{\Delta}^1_{2n+3}$ good wellorder (from [15, §3]). □

2.2. Some fine structure. In a few places, we will use hulls of the form $M = \mathrm{Hull}^P_\omega(\emptyset)$, where $P \models \mathsf{ZF}^-$ is a premouse. We now explore this a little.

2.4. Lemma. *Let P be a passive premouse.*

If P is ω-sound, then ω-maximal (putative) trees $\mathcal{T}$ on P correspond exactly to 0-maximal (putative) trees $\mathcal{U}$ on $\mathcal{J}_1(P)$, as the proof describes.

Suppose $P \models \mathsf{ZF}^-$. Then $P \preccurlyeq_1 \mathcal{J}_1(P)$. Let $M = \mathrm{Hull}^P_\omega(\emptyset)$ and $H = \mathrm{Hull}_1^{\mathcal{J}_1(P)}(\{\mathrm{OR}^P\})$. Then $H = \mathcal{J}_1(M)$, $\rho_1^H = \omega$, $p_1^H = \{\mathrm{OR}^M\}$, p_1^H is 1-solid, and H is fully sound.

Proof. For the first claim, let $\mathcal{T}$ be on P. By induction, build $\mathcal{U}$ on $\mathcal{J}_1(P)$, identical to $\mathcal{T}$, except that when $[0,\alpha]_{\mathcal{T}}$ does not drop, $M^{\mathcal{U}}_\alpha = \mathcal{J}_1(M^{\mathcal{T}}_\alpha)$. Both trees use the same functions in forming ultrapowers. We omit further details.

Now assume $P \models \mathsf{ZF}^-$. We show $P \preccurlyeq_1 \mathcal{J}_1(P)$. Let φ be a Σ_1 formula in the premouse language $\mathcal{L}$. Let $z \in P$ and $\mathcal{J}_1(P) \models \varphi(z)$. Let $\varphi_n \in \mathcal{L}$ be such that for all passive premice Q and $x \in Q$, $Q \models \varphi_n(x)$ iff $\mathcal{S}_n(Q) \models \varphi(x)$. (By [3, 1.2] this is a Σ_0 relation of (Q,x). So φ_n exists.) Now $P \models \mathsf{ZF}^-$, so if $P \models \varphi_n(z)$, so does some passive $P' \triangleleft P$, implying $\mathcal{J}_1(P') \models \varphi(z)$, so $P \models \varphi(z)$, as required.

Now we consider H. Clearly $\rho_1^H = \omega$. Let $\eta = \mathrm{OR}^P$ and $\bar\eta$ be the collapse of η to H; so $H = \mathcal{J}_1(H|\bar\eta)$. By the first claim, $H|\bar\eta \preccurlyeq_1 H$. So $\mathrm{Th}_1^H(\bar\eta) \in H$. Since $H = \mathrm{Def}_1^H(\{\bar\eta\})$, it follows that $p_1^H = \{\bar\eta\}$, H is 1-solid, 1-sound, and since $\rho_1^H = \omega$, H is fully sound. To complete the proof, $H = \mathcal{J}_1(M)$ because

$$\mathrm{Def}^P_\omega(\emptyset) = \mathrm{Def}_1^{\mathcal{J}_1(P)}(\{\eta\}) \cap P. \tag{2}$$

The $\subseteq$ direction is clear; the $\supseteq$ direction requires a $\Sigma_1^{\mathcal{J}_1(P)}(\{\eta\})$ definition to be converted to $\Sigma_0^{P\cup\{P\}}(\emptyset)$, which is similar to the argument that $P \preccurlyeq_1 \mathcal{J}_1(P)$. □

[4] A third proof for M_2 uses uniformization for Π^1_3, but one must first establish $\undertilde{\Delta}^1_2$ determinacy and Σ^1_3-correctness of M_2.

2.3. Finite support for iteration trees. The analysis of homogeneously Suslin sets will depend on understanding the component measures of a homogeneity system. We now prove a lemma toward this.

Given a normal iteration $\mathcal{T}$ on a premouse N and a measure over N derivable from $i^{\mathcal{T}}$, we will need to replace $\mathcal{T}$ with a *finite* tree, from which the same measure is derivable. This idea is straightforward and similar constructions have been given elsewhere, but here we need a little more (2.9), and we explicitly deal with the issues with type 3 extenders considered in [20] and [14, §7].

Let $\mathcal{T}$ have length $\theta+1$ and let $\mathcal{F} \subseteq M^{\mathcal{T}}_\theta$ be finite ($\mathcal{F}$ could generate the measure we're interested in). We will capture $\mathcal{F}$ by defining a finite normal tree $\mathcal{S}$ and liftup maps to $\mathcal{T}$, with $\mathcal{F}$ in the range of the ultimate liftup map. The method is straightforward: find a subset of $\mathcal{T}$ sufficient to generate $\mathcal{F}$, then perform a reverse copying construction to produce $\mathcal{S}$.

2.5. Remark. The following tool will help us handle objects in $P - \mathfrak{C}_0(P)$, when P is a type 3 premouse. The issue was ignored in the copying construction of [9]. Here it is not hard to deal with; a more general discussion is given in [14, §7].

2.6. Definition. Let P be a premouse. Define the *representation projection* function $\mathrm{rprj}^P : P \to \mathfrak{C}_0(P)$ as follows. If $x \in \mathfrak{C}_0(P)$, $\mathrm{rprj}^P(x) = x$. If $x \notin \mathfrak{C}_0(P)$, $\mathrm{rprj}^P(x)$ is the $<_P$-least pair (a,f) such that $x = [a,f]^P_{F^P}$.

Given an iteration tree $\mathcal{U}$ and $\beta < \mathrm{lh}(\mathcal{U})$, let $\mathrm{rprj}^{\mathcal{U}}_\beta = \mathrm{rprj}^{M^{\mathcal{U}}_\beta}$.

2.7. Definition (Finite Support). Let N, $\mathcal{T}$, θ and $\mathcal{F}$ be as above. A finite set $\mathbb{S}$ *supports* $\mathcal{F}$, relative to $\mathcal{T}$, given the following properties (a)-(g). Let $N_\alpha = M^{\mathcal{T}}_\alpha$ and $\mathrm{rprj}_\alpha = \mathrm{rprj}^{\mathcal{T}}_\alpha$ for $\alpha \leq \theta$.

(a) $\mathbb{S} \subseteq \{(\alpha, x) \mid \alpha \leq \theta \ \&\ x \in \mathfrak{C}_0(N_\alpha)\}$.

Let $\mathbb{S}_\alpha$ denote the section of $\mathbb{S}$ at α. Let $I \subseteq \theta+1$ be the projection of $\mathbb{S}$ on the first co-ordinate. Then

(b) $\theta \in I$; and

(c) $\mathrm{rprj}_\theta``\mathcal{F} \subseteq \mathbb{S}_\theta$.

Let $\alpha \in I$, $\alpha > 0$. There are two cases.

Case 1. α is a successor ordinal.

Let $\alpha = \beta+1$ and $\gamma = \mathcal{T}\mathrm{-pred}(\beta+1)$.

$(\mathrm{d})_{\beta+1}$ $\beta, \gamma, \gamma+1 \in I$.

Let $n = \deg^{\mathcal{T}}(\beta+1)$ and $N^* = N^*_{\beta+1} = M^{*,\mathcal{T}}_{\beta+1} \trianglelefteq N_\gamma$. Let $x \in \mathbb{S}_{\beta+1}$.

$(\mathrm{e})_{\beta+1,x}$ There's a nested n-Skolem term τ_x, parameter $q_x \in \mathfrak{C}_0(N^*)$ and $a_x \in \mathrm{lh}(E^{\mathcal{T}}_\beta)^{<\omega}$ such that $(\beta, \mathrm{rprj}_\beta(a_x)), (\gamma, \mathrm{rprj}_\gamma(q_x)) \in \mathbb{S}$ and $x = [a_x, f_{\tau_x,q_x}]^{N^*}_{E^{\mathcal{T}}_\beta}$.

$(\mathrm{f1})_{\beta+1}$ If $E^{\mathcal{T}}_\beta \in N_\beta$ then $\mathrm{rprj}_\beta(E^{\mathcal{T}}_\beta) \in \mathbb{S}_\beta$.

$(\mathrm{f2})_{\beta+1}$ If N_β is type 3 and $E^{\mathcal{T}}_\beta = F^{N_\beta}$, then $\mathrm{rprj}(\nu^{N_\beta}) \in \mathbb{S}_\beta$.

Case 2. α is a limit ordinal.

$(\mathrm{g})_\alpha$ $I \cap \alpha \neq \emptyset$, and if $\beta = \max(I \cap \alpha)$ then:

(i) $0 <_{\mathcal{T}} \beta <_{\mathcal{T}} \alpha$

(ii) $i^{\mathcal{T}}_{\beta,\alpha}$ exists and $\deg^{\mathcal{T}}(\beta) = \deg^{\mathcal{T}}(\alpha)$

(iii) $\mathbb{S}_\alpha \subseteq i^{\mathcal{T}}_{\beta,\alpha}``\mathbb{S}_\beta$
(iv) $\beta = \gamma + 1$ for some γ, and $\mathrm{crit}(i^{\mathcal{T}}_{\beta,\alpha}) > \nu(E^{\mathcal{T}}_\gamma)$.

This completes the definition of finite support. Condition $(\mathrm{g})_\alpha$(iv) helps ensure the normality of the finite tree we build.

2.8. Lemma. *Let N, $\mathcal{T}$, θ and $\mathcal{F}$ be as in 2.7. Then there is a finite support for $\mathcal{F}$ relative to $\mathcal{T}$.*

Proof. Assume $0 \in \mathcal{F}$. We recursively define finite sets $I'_i, \mathbb{S}'_i$ approximating the desired $I, \mathbb{S}$, with $I'_i \subseteq I'_{i+1}$ and $\mathbb{S}'_i \subseteq \mathbb{S}'_{i+1}$. We also define ordinals $\alpha_i \in I'_i$.

Let $I'_0 = \{\theta\}$, $\mathbb{S}'_0 = \{\theta\} \times \mathrm{rprj}_\theta``\mathcal{F}$ and $\alpha_0 = \theta$. Given $\alpha_n, I'_n, \mathbb{S}'_n$ with $\alpha_n > 0$, we process α_n, attempting to meet the requirements of 2.7 for $\alpha = \alpha_n$.

If $\alpha = \alpha_n = \beta + 1$ is a successor, let $I'_{n+1} = I'_n \cup \{\beta, \gamma, \gamma + 1\}$ (notation as in $2.7(\mathrm{d})_{\beta+1}$). Extend $\mathbb{S}'_n$ to $\mathbb{S}'_{n+1}$ by adding (finitely many) appropriate $(\beta, \mathrm{rprj}_\beta(a_x))$, $(\gamma, \mathrm{rprj}_\gamma(q_x))$, etc., to satisfy $(\mathrm{e})_{\beta+1,x}$, $(\mathrm{f1})_{\beta+1}$ and $(\mathrm{f2})_{\beta+1}$. If $\gamma + 1 < \alpha_n$, also put $(\gamma + 1, 0) \in \mathbb{S}'_{n+1}$.

If α_n is a limit, let β be least such that $\beta \geq \max(I'_n \cap \alpha_n)$, and (i),(ii) and (iv) of $(\mathrm{g})_{\alpha_n}$ are satisfied, and $(\mathbb{S}'_n)_{\alpha_n} \subseteq \mathrm{rg}(i^{\mathcal{T}}_{\beta,\alpha_n})$. (If no sufficiently large $\beta <_{\mathcal{T}} \alpha_n$ satisfied condition (iv), then $M^{\mathcal{T}}_{\alpha_n}$ would be illfounded; this is because no $E^{\mathcal{T}}_\gamma$ is of superstrong type, so if $\beta' = \mathcal{T}\text{-pred}(\gamma + 1)$ then $i^{\mathcal{T}}_{\beta',\gamma+1}(\mathrm{crit}(E^{\mathcal{T}}_\gamma)) > \nu(E^{\mathcal{T}}_\gamma)$.)

Let $I'_{n+1} = I'_n \cup \{\beta\}$. Extend $\mathbb{S}'_n$ to $\mathbb{S}'_{n+1}$ by adding all (β, x) such that $i^{\mathcal{T}}_{\beta,\alpha_n}(x) \in (\mathbb{S}'_n)_{\alpha_n}$.

In either case, set $\alpha_{n+1} = \beta = \max(I'_{n+1} \cap \alpha_n)$, completing this stage.

For some n, $\alpha_0 > \alpha_1 > \ldots > \alpha_n = 0$. Note $\{\alpha_0, \ldots, \alpha_n\} = I'_n$ and every $\alpha_i > 0$ got processed at some stage. Note for $i \leq n$, $(\mathbb{S}'_i)_{\alpha_i} = (\mathbb{S}'_n)_{\alpha_i}$. With these facts, one can check $\mathbb{S}'_n$ supports $\mathcal{F}$, and has projection I'_n. □(Lemma 2.8)

2.9. Lemma. *Let N, $\mathcal{T}$, θ and $\mathcal{F}$ be as in 2.7. Then there is a normal iteration tree $\mathcal{S}$ on N with $\mathrm{lh}(\mathcal{S}) = n + 1 < \omega$, $\deg^{\mathcal{S}}(n) = \deg^{\mathcal{T}}(\theta)$, and a near $\deg^{\mathcal{T}}(\theta)$-embedding $\bar{\varrho} : M^{\mathcal{S}}_n \to M^{\mathcal{T}}_\theta$, with $\mathrm{rprj}^{\mathcal{T}}_\theta``\mathcal{F} \subseteq \mathrm{rg}(\bar{\varrho})$. Moreover, if $i^{\mathcal{T}}$ exists then so does $i^{\mathcal{S}}$, and the embeddings commute: $i^{\mathcal{T}} = \bar{\varrho} \circ i^{\mathcal{S}}$.*

Suppose further that $\sigma_0 = \sigma : N \to N'$ is elementary, and $\sigma\mathcal{T}$ consists of wellfounded models. Let $\langle \sigma_\alpha \rangle_{\alpha \leq \theta}$ be the liftup maps from $\mathcal{T}$ to $\sigma\mathcal{T}$. Let $\langle \bar{\sigma}_i \rangle_{i \leq n}$ be the liftup maps from $\mathcal{S}$ to $\sigma\mathcal{S}$ ($\bar{\sigma}_0 = \sigma$). Then $\bar{\varrho}$ may be chosen such that there is a near $\deg^{\mathcal{T}}(\theta)$-embedding $\varrho : M^{\sigma\mathcal{S}}_n \to M^{\sigma\mathcal{T}}_\theta$ such that $\sigma_\theta \circ \bar{\varrho} = \varrho \circ \bar{\sigma}_n$, and if $i^{\mathcal{T}}$ (and therefore also $i^{\mathcal{S}}$, $i^{\sigma\mathcal{T}}$ and $i^{\sigma\mathcal{S}}$) exists then $i^{\sigma\mathcal{T}} = \varrho \circ i^{\sigma\mathcal{S}}$.

Proof. Let $\mathbb{S}$ support $\mathcal{F}$ relative to $\mathcal{T}$. We will perform a "reverse copying construction", copying down the parts of $\mathcal{T}$ appearing in $\mathbb{S}$. We use I as the index set for $\mathcal{S}$. Let $(<_{\mathcal{S}}, D^{\mathcal{S}}, \deg^{\mathcal{S}}) = (<_{\mathcal{T}}, D^{\mathcal{T}}, \deg^{\mathcal{T}}) \upharpoonright I$. Denote the models of $\mathcal{S}$ by M_α. The tree $\mathcal{S}$ we literally define will be padded. Padding occurs just at ordinals $\beta = \max(I \cap \lambda)$, where $\lambda \in I$ is a limit ordinal: for such β, λ we set $E^{\mathcal{S}}_\beta = \emptyset$, $M_\lambda = M_\beta$ and $i^{\mathcal{S}}_{\beta,\lambda} = \mathrm{id}$. (We plan for $\mathcal{S}$ to be normal. If $\lambda \in I$ is a limit ordinal and $\alpha + 1 \in I$, we adopt the "padding convention" that normality requires that $\mathcal{S}\text{-pred}(\alpha + 1) \neq \max(I \cap \lambda)$.)

We'll define near $\deg^{\mathcal{T}}(\alpha)$-embeddings $\pi_\alpha : M_\alpha \to N_\alpha$ by recursion on $\alpha \in I$. Suppose some π_α has been defined. If N_α is active, let $\psi_\alpha : \mathrm{Ult}_0(M_\alpha, F^{M_\alpha}) \to \mathrm{Ult}_0(N_\alpha, F^{N_\alpha})$ be the canonical map induced by π_α. Otherwise let $\psi_\alpha = \pi_\alpha$. In either case we therefore have $\pi_\alpha \subseteq \psi_\alpha$. During the recursion, we will maintain an induction hypothesis on α, which we call φ_α. We will establish φ_α while defining $\mathcal{S}\restriction(I\cap\alpha+1)$ and π_α (and therefore ψ_α). The hypothesis φ_α states: For all δ, γ and $\xi+1$ in $I\cap(\alpha+1)$, the following conditions hold:

- $\mathcal{S}$'s extenders: either $\psi_\xi(E^{\mathcal{S}}_\xi) = E^{\mathcal{T}}_\xi$ or else $E^{\mathcal{S}}_\xi$ and $E^{\mathcal{T}}_\xi$ are the active extenders of M_ξ and N_ξ respectively,
- ν-lh-preservation: $\psi_\xi(\nu(E^{\mathcal{S}}_\xi)) = \nu(E^{\mathcal{T}}_\xi)$ and $\psi_\xi(\mathrm{lh}(E^{\mathcal{S}}_\xi)) = \mathrm{lh}(E^{\mathcal{T}}_\xi)$,
- Strong Closeness at ξ, as in [12, 1.3],
- Elementarity: $\pi_\gamma : M_\gamma \to N_\gamma$ is a near $\deg^{\mathcal{T}}(\gamma)$-embedding,
- Range: $\mathrm{rg}(\pi_\gamma) \supseteq \mathbb{S}_\gamma$,
- Agreement: if $\xi<\gamma$ then ψ_ξ agrees with π_γ below $\mathrm{lh}(E^{\mathcal{S}}_\xi)+1$,
- Commutativity: if $\delta <_{\mathcal{S}} \gamma$ and $i^{\mathcal{S}}_{\delta,\gamma}$ is defined then $\pi_\gamma \circ i^{\mathcal{S}}_{\delta,\gamma} = i^{\mathcal{T}}_{\delta,\gamma}\circ\pi_\delta$.

Now we begin. Set $M_0 = N_0 = N$ and $\pi_0 = \mathrm{id}$; clearly φ_0 holds.

Suppose we have defined $\mathcal{S}\restriction(I\cap\beta+1)$, π_γ has been defined for $\gamma\le\beta$, φ_β holds and $\beta+1\in I$. We must define $\mathcal{S}\restriction(I\cap\beta+2)$ and $\pi_{\beta+1}$, and verify $\varphi_{\beta+1}$.

First we define $E^{\mathcal{S}}_\beta$ and show that it is indexed above $\mathcal{S}$'s earlier extenders.

If $E^{\mathcal{T}}_\beta \in \mathrm{rg}(\pi_\beta)$, set $E^{\mathcal{S}}_\beta = \pi_\beta^{-1}(E^{\mathcal{T}}_\beta)$. If $E^{\mathcal{T}}_\beta = F^{N_\beta}$, set $E^{\mathcal{S}}_\beta = F^{M_\beta}$.

Otherwise, since $\mathbb{S}_\beta \subseteq \mathrm{rg}(\pi_\beta)$, 2.7(f1)$_{\beta+1}$ implies N_β is type 3 and $\nu^{N_\beta} < \mathrm{lh}(E^{\mathcal{T}}_\beta) < \mathrm{OR}^{N_\beta}$, and letting $E = E^{\mathcal{T}}_\beta$, $\mathrm{rprj}_\beta(E) = (a_E, f_E) \in \mathrm{rg}(\pi_\beta)$. Let $(\bar{a_E}, \bar{f_E}) = \pi_\beta^{-1}((a_E,f_E))$ and $\bar{E} = [\bar{a_E},\bar{f_E}]^{M_\beta}_{F^{M_\beta}}$. So $\psi_\beta(\bar{E}) = E$ and $\bar{E}$ is on the sequence of $\mathrm{Ult}_0(M_\beta, F^{M_\beta})$. To set $E^{\mathcal{S}}_\beta = \bar{E}$ we need to verify that $\bar{E}$ is on $\mathbb{E}^{M_\beta}$.

For this, let $\nu = \nu^{N_\beta}$ and $\bar\nu = \nu^{M_\beta}$. Then ψ_β"$\bar\nu \subseteq \nu$; and $\psi_\beta(\bar\nu)\ge\nu$ since "$[a,f]$ represents a cardinal not in my OR" is Π_1 and π_β is at least that elementary. But $\mathrm{lh}(E)\in\mathrm{rg}(\psi_\beta)$ and ψ_β preserves cardinality, so in fact $\psi_\beta(\bar\nu)=\nu$ and $\psi_\beta(\mathrm{OR}^{M_\beta}) = \mathrm{OR}^{N_\beta}$, which implies $\bar\nu < \mathrm{lh}(\bar{E}) < \mathrm{OR}^{M_\beta}$, as required.

The agreement condition ensures $E^{\mathcal{S}}_\beta$ is indexed above $\mathcal{S}$'s earlier extenders.

The ν-lh-preservation for $\xi=\beta$ is as usual unless N_β is type 3 and $E^{\mathcal{T}}_\beta = F^{N_\beta}$. In this case (f2)$_{\beta+1}$ gives $a,f\in\mathfrak{C}_0(M_\beta)$ such that $\pi_\beta(a,f) = \mathrm{rprj}_\beta(\nu^{N_\beta})$. By Σ_1-elementarity, $\nu^{M_\beta} = [a,f]_{F^{M_\beta}}$. This implies "$\nu$-preservation" and thus, "lh-preservation".

One can now verify the tree, drop and degree structure required for the normality of $\mathcal{S}\restriction(I\cap\beta+2)$ matches that of $\mathcal{T}\restriction(I\cap\beta+2)$. Just a couple of remarks. Let $\gamma = \mathcal{T}\text{-pred}(\beta+1)$. We have $\gamma,\gamma+1\in I$ by 2.7(d)$_{\beta+1}$. Therefore setting $\mathcal{S}\text{-pred}(\beta+1)=\gamma$ upholds our padding convention: For any limit $\lambda\in I$, $\gamma\ne\max(I\cap\lambda)$. If there is a model-drop, we get $\psi_\gamma(M^*_{\beta+1}) = N^*_{\beta+1}$ by the usual argument but with ψ_γ replacing π_γ; in particular this works when $M^*_{\beta+1}\notin\mathfrak{C}_0(M_\gamma)$. As for degrees, π_γ being a *near* $\deg^{\mathcal{T}}(\gamma)$-embedding implies that setting $\deg^{\mathcal{S}}(\beta+1) = \deg^{\mathcal{T}}(\beta+1)$ is also as required for normality. [5]

We have that ψ_γ agrees with ψ_β below $(\kappa^+)^{M_\beta|\mathrm{lh}(E^{\mathcal{S}}_\beta)}$. Strong closeness at β now follows by inspection of the proof of closeness ([9, 6.1.5]). (In the proof

[5]If π_γ were only a weak $k=\deg^{\mathcal{T}}(\gamma)$-embedding then it would be possible to have $M^*_{\beta+1} = M_\gamma$, $\mathrm{crit}(E^{\mathcal{S}}_\beta) = \kappa \ge \rho_k^{M_\gamma}$ and $\pi_\gamma(\kappa) < \rho_k^{N_\gamma}$. This would require setting $\deg^{\mathcal{S}}(\beta+1) < k = \deg^{\mathcal{T}}(\beta+1)$, which would break the construction. See [14, §7] for an example.

of Claim 2 of that lemma, one must show that various maps π_δ preserve Σ_1 definitions of measures, working backwards through ordinals δ. One uses the inductive commutativity hypothesis in propagating this preservation through a padded stage of $\mathcal{S}$.)

So the hypotheses for the Shift Lemma hold (with the appropriate initial segments $M^*_{\beta+1}, N^*_{\beta+1}$ of M_γ, N_γ and degree $m = \deg^{\mathcal{T}}(\beta+1) = \deg^{\mathcal{S}}(\beta+1)$), yielding $\pi_{\beta+1}$ in the usual way:

$$\pi_{\beta+1}([a, f_{\tau,q}]^{M^*_{\beta+1}}_{E^{\mathcal{S}}_\beta}) = [\psi_\beta(a), f_{\tau,\psi_\gamma(q)}]^{N^*_{\beta+1}}_{E^{\mathcal{T}}_\beta} \tag{3}$$

We use ψ_β, ψ_γ here instead of π_β, π_γ in case we have $E^{\mathcal{T}}_\beta \in N_\beta - \mathfrak{C}_0(N_\beta)$, or $N^*_{\beta+1} \in N_\gamma - \mathfrak{C}_0(N_\gamma)$. As π_γ is a near $\deg^{\mathcal{S}}(\gamma)$-embedding, the Shift Lemma, strong closeness at β and the argument from [12, 1.3] give that $\pi_{\beta+1}$ is a near $\deg^{\mathcal{S}}(\beta+1)$-embedding. The Shift Lemma gives commutativity, and its proof shows $\pi_{\beta+1}$ agrees with ψ_β below $\mathrm{lh}(E^{\mathcal{S}}_\beta)+1$, implying the agreement condition.

Finally we show $\mathrm{rg}(\pi_{\beta+1}) \supseteq \mathbb{S}_{\beta+1}$. Let $x \in \mathbb{S}_{\beta+1}$, and let a_x, q_x, τ_x be as in 2.7(e)$_{\beta+1,x}$. By φ_β, we have $\mathrm{rprj}_\beta(a_x) \in \mathrm{rg}(\pi_\beta)$ and $\mathrm{rprj}_\gamma(q_x) \in \mathrm{rg}(\pi_\gamma)$. This implies that $a_x \in \mathrm{rg}(\psi_\beta)$ and $q_x \in \mathrm{rg}(\psi_\gamma)$; denote the preimages $\bar{a}, \bar{q}$. Note $\bar{q} \in M^*_{\beta+1}$ and $\bar{a} \subseteq \mathrm{lh}(E^{\mathcal{S}}_\beta)$ and $\deg^{\mathcal{S}}(\beta+1) = m = \deg^{\mathcal{T}}(\beta+1)$. So $\bar{x} = [\bar{a}, f_{\tau_x,\bar{q}}]$ is an element of $M_{\beta+1} = \mathrm{Ult}_m(M^*_{\beta+1}, E^{\mathcal{S}}_\beta)$. By (3), $\pi_{\beta+1}(\bar{x}) = x$, as required.

This establishes $\varphi_{\beta+1}$.

Now suppose we have $\mathcal{S} \restriction (I \cap \beta+1)$ for some $\beta \in I$, $\beta < \theta$, and φ_β holds, but $\beta+1 \notin I$. So $\inf(I - (\beta+1))$ is a limit α. Thus we set $E^{\mathcal{S}}_\beta = \emptyset$ and $M_\alpha = M_\beta$. Let $\pi_\alpha = i^{\mathcal{T}}_{\beta,\alpha} \circ \pi_\beta$. This yields a near $\deg^{\mathcal{T}}(\alpha) = \deg^{\mathcal{S}}(\alpha)$-embedding since there is no dropping of any kind in $(\beta,\alpha]_{\mathcal{T}}$. Its range is large enough since $\mathrm{rg}(\pi_\beta) \supseteq \mathbb{S}_\beta$ and $i^{\mathcal{T}}_{\beta,\alpha}$"$\mathbb{S}_\beta \supseteq \mathbb{S}_\alpha$. By 2.7(g)$_\alpha$(iv), β is a successor, and

$$\pi_\beta(\nu(E^{\mathcal{S}}_{\beta-1})) = \nu(E^{\mathcal{T}}_{\beta-1}) < \mathrm{crit}(i^{\mathcal{T}}_{\beta,\alpha}).$$

(The equality is by ν-lh-preservation and agreement.) Therefore $i^{\mathcal{T}}_{\beta,\alpha} \circ \pi_\beta$ agrees with π_β below $\mathrm{lh}(E^{\mathcal{S}}_{\beta-1})+1$, giving agreement. Commutativity is clear. This shows φ_α, completing the induction.

Setting $\bar{\varrho} = \pi_\theta$, the first part of the theorem has been proven. With this choice of $\bar{\varrho}$, we just sketch the second part.

First, for P, Q type 3 premice and $\sigma' : \mathfrak{C}_0(P) \to \mathfrak{C}_0(Q)$ a weak 0-embedding, let $\psi_{\sigma'} : \mathrm{Ult}_0(P, F^P) \to \mathrm{Ult}_0(Q, F^Q)$ be induced by σ'. Suppose $\psi_{\sigma'}(\nu^P) = \nu^Q$. Then if $x \in P$, then $\psi_{\sigma'}(x) \in Q$ and

$$\psi_{\sigma'}(\mathrm{rprj}^P(x)) = \mathrm{rprj}^Q(\psi_{\sigma'}(x)). \tag{4}$$

Now $\sigma\mathcal{T}$ has exactly the same structure (nodes, drops and degrees) as $\mathcal{T}$. (I.e., the problems with type 3 extenders and degree differences discussed in [14, §7] and [20, §4.1] don't occur.) For [14, 7.1] and [12, 1.3] apply since σ is fully elementary. In particular, for $\alpha \le \theta$, σ_α is a near $\deg^{\mathcal{T}}(\alpha)$-embedding (cf. footnote 5), and if $M^{\mathcal{T}}_\alpha$ is type 3, then $\psi_{\sigma_\alpha}(\nu^{M^{\mathcal{T}}_\alpha}) = \nu^{M^{\sigma\mathcal{T}}_\alpha}$. Therefore (4) applies when $\sigma' = \sigma_\alpha$. We have an analogous situation with $\sigma\mathcal{S}$ and $\mathcal{S}$. Let $\bar{\sigma}_\alpha$ be the maps lifting $\mathcal{S}$ to $\sigma\mathcal{S}$ (for $\alpha \in I$).

Now if $(\mathbb{S}')_\beta = \sigma_\beta``\mathbb{S}_\beta$ for all $\beta \leq \theta$, then $\mathbb{S}'$ supports $\mathcal{F}' = \sigma_\theta``(\mathrm{rprj}^{\mathcal{T}}_\theta``\mathcal{F})$. (In fact, $\mathbb{S}'$ is produced by the algorithm of Lemma 2.8, applied to $\sigma\mathcal{T}$ and $\mathcal{F}'$. To see this use the facts from the previous two paragraphs.)

Let $\mathcal{S}'$ be the finite tree obtained by the method of the first part of the theorem, applied to $\sigma\mathcal{T}$ and $\mathbb{S}'$, and let $\pi'_\alpha : M^{\mathcal{S}'}_\alpha \to M^{\sigma\mathcal{T}}_\alpha$ be the lifting maps (for $\alpha \in I$). One shows inductively that $\mathcal{S}' \restriction (\alpha+1) = \sigma\mathcal{S} \restriction (\alpha+1)$ and the commutativity $\sigma_\alpha \circ \pi_\alpha = \pi'_\alpha \circ \bar{\sigma}_\alpha$. The desired ϱ is π'_θ. □(Lemma 2.9)

2.10. Definition. Let $\mathcal{T}, \mathcal{F}, \mathbb{S}$ be as in 2.7. The finite support tree $\mathcal{T}^{\mathbb{S}}_{\mathcal{F}}$ for $\mathcal{F}$, relative to $\mathcal{T}, \mathbb{S}$, is the tree $\mathcal{S}$ as defined in the proof of 2.9. If $\mathbb{S}^*$ is the support for $\mathcal{F}$ defined in the proof of 2.8, the finite support tree $\mathcal{T}_{\mathcal{F}}$ is $\mathcal{T}^{\mathbb{S}^*}_{\mathcal{F}}$.

§3. Homogeneously Suslin sets in M_n. We are now ready for the main argument. Let N be a mouse modelling ZF^-. Suppose that in N, T is a homogeneous tree and we want to bound the descriptive complexity of $p[T]$. Let $\nu = \langle \nu_s \rangle_{s \in {}^{<\omega}\omega} \in N$ be an homogeneity system for T. For $x \in \mathbb{R}$, let ν_x be the tower $\left\langle \nu_{x\restriction n}; i^\nu_{x\restriction n, x\restriction m} \right\rangle_{n \leq m < \omega}$. For any tower μ on N, let

$$U^N_\mu = \mathrm{Ult}(N, \mu) = \mathrm{dirlim}_{m \leq n < \omega}(\mathrm{Ult}(N, \mu_n); i^\mu_{m,n}).$$

Let $U_x = U^N_{\nu_x}$. So if $x \in N$ then $N \models$ "$x \in p[T]$ iff U_x is wellfounded". We would like to replace "U_x is wellfounded" in this statement with some formula $\varphi(x)$, where (in the case of M_n) φ is projective. We will have φ assert the iterability of an associated countable premouse. So suppose $\pi : M \to N$ is elementary with $\nu \in \mathrm{rg}(\pi)$. Let $\pi(\bar{\nu}) = \nu$. Let $\bar{U}_x = U^M_{\bar{\nu}_x}$. Our plan will be to show that for reals $x \in N$, U_x is wellfounded iff $\bar{U}_x$ is iterable. Given that, the complexity of "$\bar{U}_x$ is iterable" will bound the complexity of $p[T]$. The main issue is to show that the iterability of $\bar{U}_x$ implies the wellfoundedness of U_x. We deal with that now.

3.1. Lemma. *Let N be a mouse modelling ZF^- and $\pi : M \to N$ be elementary. Let Σ be the iteration strategy for M induced by π and some fixed strategy for N. Let $\bar{\mu} \subseteq M$ be a tower of measures on M and let $\bar{U} = \mathrm{Ult}(M, \bar{\mu})$.*

Suppose $\bar{\mathcal{T}}$ is a normal tree on M, via Σ, with last model $\bar{Q}$, $i^{\bar{\mathcal{T}}}$ exists, and $\sigma : \bar{U} \to Q$ is elementary and such that $\sigma \circ i^M_{\bar{\mu}} = i^{\bar{\mathcal{T}}}$.

Then $U = \mathrm{Ult}(N, \pi``\bar{\mu})$ is wellfounded; in fact there is $\psi : U \to Q$, with ψ elementary, and Q the last model of $\pi\bar{\mathcal{T}}$.

Proof. We first give the proof assuming that $(*)$ if M has a largest cardinal θ, then $\mathrm{cof}(\theta)^M$ isn't measurable in M.

Let's set up notation for various natural maps. See Figure 1 for the main ones.

We assume μ_0 is trivial. Let $j_{n,\infty} : \mathrm{Ult}(N, \mu_n) \to U$ be the canonical map and $j = j_{0,\infty} : N \to U$. Define $\bar{j}_{n,\infty}$ and $\bar{j}$ analogously at the M level. The Shift Lemma applied to π and $\bar{\mu}_n$ gives $\pi \restriction \mathrm{Ult}(M, \bar{\mu}_n) : \mathrm{Ult}(M, \bar{\mu}_n) \to \mathrm{Ult}(N, \mu_n)$ (this uses $(*)$). Let $\pi_\infty : \bar{U} \to U$ be the unique embedding commuting with these maps.

Let $\mathcal{T} = \pi\bar{\mathcal{T}}$ be the copied tree on N. Let $\pi_{\bar{Q}}$ be the final copy map. Then ignoring ψ, Figure 1 shows a commuting diagram of elementary maps.

We now want to define $\psi : U \to Q$ in the only elementary, commuting way:

$$\psi\big(j(f)(\pi_\infty(b))\big) = i^{\mathcal{T}}(f)\big(\pi_{\bar{Q}} \circ \sigma(b)\big). \tag{5}$$

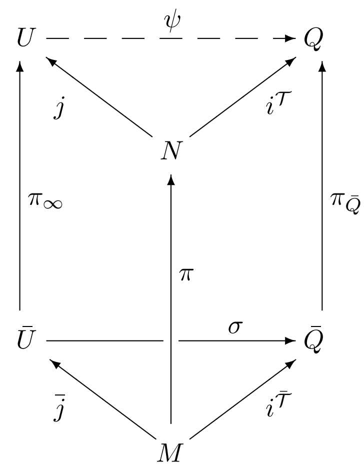

Figure 1. Commuting diagram.

(For all $x \in U$ there is $f \in N$ and $b \in \bar{U}$ such that $x = j(f)(\pi_\infty(b))$. For there is $f \in N$ and $n < \omega$ such that $x = j(f)(j_{n,\infty}(a'))$, where a' is the generator for μ_n. But $\pi(\bar{\mu}_n) = \mu_n$, so if a is the generator for $\bar{\mu}_n$ then $\pi(a) = a'$. By commutativity, f and $b = \bar{j}_{n,\infty}(a)$ work.)

We need to see that ψ is well-defined and elementary. This requires certain measures derived from j and $i^{\mathcal{T}}$ to be identical.

Conventions & Notation. Let $k : P \to R$ be elementary between premice, and $x \in (k(\alpha))^{<\omega}$, with α least such. Then μ^k_x denotes the P-ultrafilter on $\alpha^{|x|}$ derived from k with generator x. Also let $\mu^{\mathcal{W}}_x = \mu^k_x$ when $\mathcal{W}$ is a normal iteration and $k = i^{\mathcal{W}}$.

To see that (5) defines an elementary embedding, it suffices to show that

$$\mu^{\mathcal{T}}_{\pi_{\bar{Q}}(\sigma(b))} = \mu^j_{\pi_\infty(b)}. \tag{6}$$

We may assume that $b = \bar{j}_{n,\infty}(a)$, where a is the generator of $\bar{\mu}_n$. This means

$$\mu^j_{\pi_\infty(b)} = \pi(\bar{\mu}_n) = \mu_n. \tag{7}$$

Now since the bottom triangle of Figure 1 commutes, $\bar{\mu}_n = \mu^{\bar{\mathcal{T}}}_{\sigma(b)}$. Let $\bar{R}$ be the last model of $\bar{\mathcal{T}}_{\sigma(b)}$ (the finite support tree, as in 2.10) and $\bar{\varrho} : \bar{R} \to \bar{Q}$ a map as given by 2.9. So $i^{\bar{\mathcal{T}}_{\sigma(b)}}$ exists and $\bar{\varrho} \circ i^{\bar{\mathcal{T}}_{\sigma(b)}} = i^{\bar{\mathcal{T}}}$, and therefore

$$\bar{\mu}_n = \mu^{\bar{\mathcal{T}}_{\sigma(b)}}_{\bar{\varrho}^{-1}(\sigma(b))}. \tag{8}$$

Since $\bar{\mathcal{T}}_{\sigma(b)}$ is a finite tree on M, it is a definable class of M, and $\pi(\bar{\mathcal{T}}_{\sigma(b)}) = \pi\bar{\mathcal{T}}_{\sigma(b)}$, and the copy maps lifting $\bar{\mathcal{T}}_{\sigma(b)}$ to $\pi\bar{\mathcal{T}}_{\sigma(b)}$ are all restrictions of π. Let R be the last model of $\pi\bar{\mathcal{T}}_{\sigma(b)}$. The second part of 2.9 gives an elementary $\varrho : R \to Q$ commuting with all maps (given a good $\bar{\varrho}$). Combining this with (8) gives

$$\mu^{\mathcal{T}}_{\pi_{\bar{Q}}(\sigma(b))} = \mu^{\pi\bar{\mathcal{T}}_{\sigma(b)}}_{\varrho^{-1}(\pi_{\bar{Q}}(\sigma(b)))} = \mu^{\pi(\bar{\mathcal{T}}_{\sigma(b)})}_{\pi(\bar{\varrho}^{-1}(\sigma(b)))} = \pi(\mu^{\bar{\mathcal{T}}_{\sigma(b)}}_{\bar{\varrho}^{-1}(\sigma(b))}) = \pi(\bar{\mu}_n) = \mu_n. \tag{9}$$

So (9) and (7) yield (6), completing the proof assuming $(*)$.

Without $(*)$, $\bar{\mathcal{T}}_{\sigma(b)}$ needn't be a class of M; in fact, its models might have height $> \mathrm{OR}^M$. Moreover, the copy maps (from $\bar{\mathcal{T}}_{\sigma(b)}$ to $\pi\bar{\mathcal{T}}_{\sigma(b)}$) needn't be restrictions of π. In this case we simply take the various ultrapowers at the "representation level", without transitivizing; e.g., the first ultrapower in $\bar{\mathcal{T}}_{\sigma(b)}$ consists of pairs $(a, f) \in M$. This version of $\bar{\mathcal{T}}_{\sigma(b)}$ *is* a class of M. Likewise with $\mathrm{Ult}(M, \bar{\mu}_n)$. The proof adapts in a straightforward way. □

3.2. Theorem. *In M_n, every homogeneously Suslin set of reals is correctly $\utilde{\Delta}^1_{n+1}$.*

Before giving the proof, we state some corollaries:

3.3. Corollary (Steel, Sargsyan, S.). *In M_n, every hom set of reals is $(\delta_0^{M_n}+1)$-universally Baire.*

Proof. By 3.2 and 4.1. □

3.4. Corollary. *In M_n, the weakly homogeneously Suslin sets of reals are precisely the $\utilde{\Sigma}^1_{n+1}$ sets.*

Proof. Weakly hom sets are just projections of hom sets. So 3.2 implies that in M_n, all weakly hom sets are $\utilde{\Sigma}^1_{n+1}$. The converse follows Martin-Steel [8]: by this, in M_n all $\utilde{\Pi}^1_n$ sets are hom. □

Proof of Theorem 3.2. If (a) is false, then in M_n, let A be $<_{M_n}$-least such that A is hom but not correctly $\utilde{\Delta}^1_{n+1}$. Let η be the first indiscernible of M_n and let $M = \mathrm{Hull}_\omega^{M_n|\eta}(\emptyset)$. So $M \in M_n$. We will show that in M_n, A is correctly $\Delta^1_{n+1}(M)$, a contradiction.

Let $\pi : M \to M_n|\eta$ be the inverse of the collapse map. Then π is elementary, so M is (ω, ω_1+1)-iterable, via the strategy Σ induced by lifting trees to $M_n|\eta$. By 2.4, we can consider Σ a $(0, \omega_1+1)$-strategy for $\mathcal{J}_1(M)$. Also by 2.4, $\mathcal{J}_1(M)$ is an ω-mouse with $\rho_1 = \omega$, so Σ is the unique such strategy for $\mathcal{J}_1(M)$.

Note $A \in \mathrm{rg}(\pi)$, so there are $T, \nu \in \mathrm{rg}(\pi)$ such that in M_n, $p[T] = A$ and ν is a homogeneity system for T. Let $\pi(\bar{\nu}) = \nu$. Applying the following claim to $\bar{\mu} = \bar{\nu}_x$ shows that if $x \in \mathbb{R}^{M_n}$, then $x \in p[T]$ iff $\mathrm{Ult}(M, \bar{\nu}_x)$ is (ω_1+1)-iterable.

Claim 1. *Let $\bar{\mu} \subseteq M$ be a tower of measures on M. Let $\mu = \pi``\bar{\mu}$. If $\bar{\mu} \in M_n$, the following are equivalent in V:* [6]

(a) *$U = \mathrm{Ult}(M_n, \mu)$ is wellfounded;*
(b) *There is an elementary $\psi : \bar{U} \to M_n|\eta$;*
(c) *$\bar{U} = \mathrm{Ult}(M, \bar{\mu})$ is (ω_1+1)-iterable;*
(d) *There is a countable Σ-iterate Q of M, and an elementary $\sigma : \bar{U} \to Q$.*

Moreover, conditions (c) and (d) are equivalent for any tower $\bar{\mu}$ on M, $\bar{\mu} \in V$.

Proof. We adopt the notation introduced prior to and during 3.1.

For "(a) ⇒ (b)", note $j(\eta) = \eta$ (as $\bar{\mu} \in M_n$). So $\pi_\infty : \bar{U} \to U|\eta$ is elementary. But $j(\bar{U}) = \bar{U} \in U$, so by absoluteness, in U there is an elementary $\pi' : \bar{U} \to U|\eta$. Pulling back under j, in M_n there is an elementary $\psi : \bar{U} \to M_n|\eta$. [7]

"(b) ⇒ (c)" is clear. For any $\bar{\mu} \in V$, we show "(c)⇒ (d)". Condition (c) and 2.4 yield a successful comparison $(\bar{\mathcal{T}}, \bar{\mathcal{U}})$ of $(\mathcal{J}_1(M), \mathcal{J}_1(\bar{U}))$, with 0-maximal

[6]The argument shows they're also equivalent inside M_n.

[7]Thanks to the referee for providing a correction to our original argument here.

trees, and $\bar{\mathcal{T}}$ via "Σ". Now $\mathcal{J}_1(M)$ is an ω-mouse with $\rho_1 = \omega$ and is the core of $\mathcal{J}_1(\bar{U})$, and $\mathcal{J}_1(\bar{U})$ is 1-solid. Standard fine structure therefore shows that $\bar{\mathcal{T}}$ and $\bar{\mathcal{U}}$ have a common final model, $i^{\bar{\mathcal{T}}}$ and $i^{\bar{\mathcal{U}}}$ exist and $i^{\bar{\mathcal{T}}} = i^{\bar{\mathcal{U}}} \circ i_{\bar{\mu}}$. Therefore $\sigma = i^{\bar{\mathcal{U}}} \upharpoonright \bar{U}$ witnesses (d).

"(d) $\Rightarrow$ (a)" is by 3.1 ($\sigma \circ \bar{j} = i^{\bar{\mathcal{T}}}$ because $M = \mathrm{Hull}_\omega^M(\emptyset)$).

For any $\bar{\mu} \in V$, "(d) $\Rightarrow$ (c)" is because the uniqueness of Σ implies that it actually follows the first round of an $(\omega, \omega_1, \omega_1+1)$-strategy. □ (Claim 1)

We now discuss the descriptive complexity of conditions (c) and (d) of Claim 1. The analysis is taken from [15], with some simple adaptations. We start with $M_n = M_1$. In this case, (d) is $\Sigma^1_2(M)$, since by 1-smallness and Q-structure arguments, Σ always chooses the unique cofinal wellfounded branch. We will now observe that (c) is $\Pi^1_2(M)$.

The following is essentially the Π_1^{HC}-iterability of [15, 1.6], but we allow unsound premice P, and have dropped the clause concerning iterates of $M_b^{\mathcal{T}}$.

3.5. Definition. A countable premouse P is Π^1_2-*iterable above* δ iff for every triple $(\alpha, \mathcal{T}, x)$ with α a countable ordinal, $\mathcal{T}$ a countable putative normal iteration tree on P, and $\mathcal{T}$ above δ: either (1) there is a $\mathcal{T}$-maximal branch b such that $M_b^{\mathcal{T}}$ is α-wellfounded; or (2) $\mathcal{T}$ has a final model, which is α-wellfounded.

P is Π^1_2-*iterable* iff it is Π^1_2-iterable above 0.

Π^1_2-iterability above δ is $\Pi^1_2(\delta)$ when coded over $\mathbb{R}$. Note that in the proof of the following claim, all Q-structures here have form $L_\gamma[M(\mathcal{T})]$, by 1-smallness.

Claim 2. *Let $n = 1$. For all towers $\bar{\mu}$ on M, $\bar{U} = \mathrm{Ult}(M, \bar{\mu})$ is (i) (ω_1+1)-iterable iff (ii) Π^1_2-iterable.*

Proof. As in [7, 6.13] or [15, 2.2(3)]. To prepare for discussing M_2 though, we give the main ideas for "(ii) $\Rightarrow$ (i)". It suffices to successfully compare $\mathcal{J}_1(M)$ with $\mathcal{J}_1(\bar{U})$. We leave the reader to synthesize the following terse remarks into such a comparison.

Let Γ be the partial strategy for $\mathcal{J}_1(\bar{U})$ defined by "Given $\mathcal{U}$ on $\mathcal{J}_1(\bar{U})$ of limit length, let $\Gamma(\mathcal{U})$ be the unique c with $Q(c,\mathcal{U})$ wellfounded". Let $(\mathcal{T},\mathcal{U})$ be a partial comparison of $(\mathcal{J}_1(M), \mathcal{J}_1(\bar{U}))$ via (Σ, Γ), of limit length $\lambda \leq \omega_1$.

If $\lambda < \omega_1$, let $b = \Sigma(\mathcal{T})$ and $Q = Q(b, \mathcal{T})$. Let $\alpha, \alpha' \geq \mathrm{OR}^Q$ and c, c' be branches witnessing the Π^1_2-iterability of $\bar{U}$ with respect to $\mathcal{U}, \alpha$ and $\mathcal{U}, \alpha'$ respectively. By the uniqueness of Γ's branch choices, c and c' are cofinal in $\mathcal{U}$.

If Q is ω-sound, 1-smallness implies $Q \triangleleft M_c^{\mathcal{U}}$ and $Q \triangleleft M_{c'}^{\mathcal{U}}$. So $c = c'$ as Q is a Q-structure. The Π^1_2-iterability of $\bar{U}$ then implies $M_c^{\mathcal{U}}$ is wellfounded (by letting $\alpha \to \omega_1$). So $\mathcal{U} ^\frown c$ is via Γ. If Q is unsound, then $(\mathcal{T} ^\frown b, \mathcal{U} ^\frown c)$ is successful. [8]

If $\lambda = \omega_1$, similar arguments, and the homogeneity of $\mathrm{Col}(\omega, \alpha)$, show $(\mathcal{T},\mathcal{U}) \in L[M, \bar{\mu}]$. But $\omega_1^{L[M,\bar{\mu}]} < \mathrm{lh}(\mathcal{T},\mathcal{U})$, contradicting comparison. □ (Claim 2)

Since the map $x \mapsto \bar{\mu} = \bar{\nu}_x$ is continuous, conditions (c) and (d) provide an M_1-correctly-$\Delta^1_2(M)$ definition of $p[T]^{M_1}$, by Claims 1 and 2 and the comments following Claim 1. [9] This completes the M_1 case.

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[8] An argument like for Claim 3 below shows that $c = c'$, even if Q is unsound, but we don't need that. Moreover, if $\bar{\mu}$ is bounded in δ^M, then Q is always ω-sound, like in 5.1(b).

[9] Conditions (a)-(d) are also equivalent *in* M_1. So the $\Delta^1_2(M)$ definition we gave has the same interpretation in M_1 as in V.

Now consider M_2. The analysis below is also from [15]. The definition of Π^1_3-iterability is a simplification of the Π_2^{HC}-iterability of [15, 1.4], except that we will need to apply it to unsound structures.

Let P be a 2-small ω-mouse and let Σ_P be its unique (ω,ω_1+1)-strategy. Given a limit length $\mathcal{T}$ via Σ_P, $b=\Sigma_P(\mathcal{T})$ is the unique $\mathcal{T}$-cofinal branch b' such that $Q(b',\mathcal{T})$ is Π^1_2-iterable above $\delta(\mathcal{T})$. For let $b'\neq b$ be such. By 2-smallness, any partial comparison of $Q(b,\mathcal{T})$ vs $Q(b',\mathcal{T})$ is above $\delta(\mathcal{T})$. So by the proof of Claim 2, there is a successful comparison. Since the Q-structures are $\delta(\mathcal{T})$-sound, fine structure then shows they're equal, and therefore that $b=b'$.

Therefore the statement "$\mathcal{T}$ is a normal tree on P via Σ_P" is $\Pi^1_2(P)$, and $\Sigma_P(\mathcal{T})$ is $\Delta^1_3(\mathcal{T})$. Applying this to M,Σ, we have that condition (d) is $\Sigma^1_3(M)$. We now observe, in two ways, that condition (c) is $\Pi^1_3(M)$.

If $\mathcal{T}$ is a putative normal iteration tree on a 2-small premouse and b is a $\mathcal{T}$-cofinal branch, let's say b is *$\mathcal{T}$-good* iff $M^{\mathcal{T}}_b$ is wellfounded and $Q(b,\mathcal{T})$ is Π^1_2-iterable above $\delta(\mathcal{T})$. Also, $\mathcal{T}$ is *good* if for each limit $\lambda<\mathrm{lh}(\mathcal{T})$, the branch $[0,\lambda]_{\mathcal{T}}$ is $\mathcal{T}\restriction\lambda$-good. Likewise for a (putative, partial) comparison.

3.6. Definition. Assume $\utilde{\Delta}^1_2$-determinacy and let P be a 2-small, k-sound premouse, with $\rho^P_{k+1}\leq\delta$. P is *Π^1_3-iterable above δ* iff for each countable good k-maximal above-δ putative tree $\mathcal{T}$ on P: either (1) $\mathcal{T}$ has a wellfounded last model; or (2) $\mathcal{T}$ has limit length and there is a $\mathcal{T}$-good branch b in $\Delta^1_3(\mathcal{T})$.

P is Π^1_3-iterable iff it is Π^1_3-iterable above 0.

By [10], $\utilde{\Delta}^1_2$-determinacy implies that $\Pi^1_3(x)$ is closed under "$\exists b\in\Delta^1_3(x)$". Therefore "$\Pi^1_3$-iterability above δ" is indeed a $\Pi^1_3(\delta)$ condition. Note we haven't required P be δ-sound. The fine-structural assumptions on P do imply that the Q-structure $Q(b,\mathcal{T})$ exists for any b (but may be illfounded). The following notion provides an alternative $\Pi^1_3(M)$ description of condition (c).

3.7. Definition. Let $\bar{\mu}$ be a tower on M and $\bar{U}=U^M_{\bar{\mu}}$. Then $\bar{U}$ is *Π^1_3-M-comparable* iff for each countable good putative partial comparison $(\mathcal{T},\mathcal{U})$ of $\mathcal{J}_1(M)$ vs $\mathcal{J}_1(\bar{U})$: either (1) $\mathcal{U}$ has a wellfounded last model; or (2) $(\mathcal{T},\mathcal{U})$ has limit length and there are $b,c\in\Delta^1_3(\mathcal{T})$ such that either (2a) c is $\mathcal{U}$-good, or (2b) b is $\mathcal{T}$-good and there is a normal tree $\mathcal{T}'$ on $Q(b,\mathcal{T})$, above $\delta(\mathcal{T})$, with last model Q, and there is a Σ_1-elementary $\sigma:\mathcal{J}_1(\bar{U})\to Q$.

Our original proof just used Π^1_3-M-comparability, which is ostensibly weaker than Π^1_3-iterability. Steel noticed that if $\mathrm{Ult}(M,\bar{\mu})$ is iterable, then it is in fact Π^1_3-iterable, and provided the proof of this, for which, the next fact is the key.[10]

3.8. Fact (Martin-Steel). *Let $\mathcal{T}$ be a normal iteration tree of limit length on a premouse P, with cofinal branches $b\neq c$ (maybe illfounded). Let $\beta\in b$, $\gamma\in c$ be such that $i^{\mathcal{T}}_{\beta,b}$ and $i^{\mathcal{T}}_{\gamma,c}$ exist. Then $\mathrm{rg}(i^{\mathcal{T}}_{\beta,b})\cap\mathrm{rg}(i^{\mathcal{T}}_{\gamma,c})\cap\delta(\mathcal{T})$ is bounded in $\delta(\mathcal{T})$.*

Proof. Use the zipper diagram of the Branch Uniqueness proof, [20, §6]. □

[10] Claim 3 (in particular, Π^1_3-iterability) gives a little more information than we need to prove 3.2. If $\bar{\mu}$ is bounded in δ^M_0, as it is if $\bar{\mu}=\bar{\nu}_x$, then the proof of 5.1 shows that the situation in which the lemma is needed never occurs in comparing M with $\bar{U}$.

Claim 3. *Let $n = 2$, let $\bar{\mu}$ be a tower on M, and $\bar{U} = \mathrm{Ult}(M, \bar{\mu})$. Then $\bar{U}$ is (i) (ω_1+1)-iterable iff (ii) Π^1_3-iterable iff (iii) Π^1_3-M-comparable.*

Proof. We just prove "(i) ⇔ (ii)". Since $\mathcal{J}_1(\bar{U})$ is not 1-sound, we can't quite quote [15]. The following proves "(i) ⇒ (ii)".

Subclaim 1. Assume that Γ is an (ω_1+1)-strategy for $\mathcal{J}_1(\bar{U})$. Let $\mathcal{U}$ be a normal tree of limit length on $\mathcal{J}_1(\bar{U})$, via Γ, and $c = \Gamma(\mathcal{U})$. Then $c \in \Delta^1_3(\mathcal{U})$; in fact, c is the unique c' such that $Q(c',\mathcal{U})$ is Π^1_2-iterable above $\delta(\mathcal{U})$.

Proof. Everything works as in the discussion preceding 3.6, except when c does not drop, and $i^{\mathcal{U}}_c(\delta^{\bar{U}}_i) = \delta(\mathcal{U})$ with $i = 0$ or $i = 1$. In this case, $Q = Q(c,\mathcal{U}) = M^{\mathcal{U}}_c$ is ω_1+1-iterable above $\delta(\mathcal{U})$. Suppose $c \neq c'$ and $Q' = Q(c',\mathcal{U})$ is Π^1_2-iterable above $\delta(\mathcal{U})$. As before, there's a successful comparison $(\mathcal{V},\mathcal{W})$ of Q vs Q', above $\delta(\mathcal{U})$, with common last model R, and $i^{\mathcal{V}}$ and $i^{\mathcal{W}}$ exist. This implies c' did not drop (since c didn't). Let $\bar{j} : \mathcal{J}_1(M) \to \mathcal{J}_1(\bar{U})$ be the unique Σ_1-elementary map. We get a Σ_1-elementary $k : \mathcal{J}_1(M) \to R$ defined two ways:

$$k = i^{\mathcal{V}} \circ i^{\mathcal{U}}_c \circ \bar{j} = i^{\mathcal{W}} \circ i^{\mathcal{U}}_{c'} \circ \bar{j}.$$

Note k is continuous at δ^M_i since it's composed of ultrapower embeddings of degree 0 and δ^M_i is regular in $\mathcal{J}_1(M)$. But $i^{\mathcal{V}}$ and $i^{\mathcal{W}}$ have critical points $> \delta(\mathcal{U})$. So $k``\delta^M_i$ is cofinal in $k(\delta^M_i) = \delta(\mathcal{U})$ and $k``\delta^M_i \subseteq \mathrm{rg}(i^{\mathcal{U}}_c) \cap \mathrm{rg}(i^{\mathcal{U}}_{c'})$, contradicting 3.8. □(Subclaim 1)

The proof of "(ii) ⇒ (i)" is essentially as in [15, 2.2(3)], with one very small addition. Note that Π^1_3-iterability defines an iteration quasi-strategy Γ for $\bar{U}$. Define a comparison of M vs $\bar{U}$, via (Σ,Γ). Given a resulting partial stage $(\mathcal{T},\mathcal{U})$, attempt to show that if $b = \Sigma(\mathcal{T})$ and c is a $\mathcal{U}$-good branch then $Q(c,\mathcal{U}) = Q(b,\mathcal{T})$, so $c \in \Delta^1_3(\mathcal{T},\mathcal{U})$, uniformly in $(\mathcal{T},\mathcal{U})$. If this fails, show b is non-dropping and $i^{\mathcal{T}}(\delta^M_i) = \delta(\mathcal{U})$ ($i = 0$ or 1), [11] choose any $\mathcal{U}$-good c and note that $M^{\mathcal{T}}_b$ and $M^{\mathcal{U}}_c$ can be successfully compared above $\delta(\mathcal{T})$. If it never fails, and the comparison lasts through ω_1 stages, obtain a contradiction like at the end of the proof of Claim 2, by applying 2.1 to show $(\mathcal{T},\mathcal{U}) \in M_1(M,\bar{\mu})$. □ (Claim 3)

By Claim 3, condition (c) is $\Pi^1_3(M)$. We saw that (d) is $\Sigma^1_3(M)$ (just prior to 3.6). By 2.1, the hom set is correctly $\utilde{\Delta}^1_3$ in M_2.

For M_3, the argument is like that for M_2, except that the definition of Π^1_4-iterability is significantly different to that of Π^1_3-iterability because of periodicity. See [15] for the elegant solution. The arguments for higher M_{2n} and M_{2n+1} are modelled on those for M_2 and M_3, respectively. □(Theorem 3.2)

Theorem 3.2 leaves the following questions unanswered. As far as the author knows, the answers might both be "no", simultaneously.

3.9. Question.

- In M_n, are all homogeneously Suslin sets $\utilde{\Pi}^1_n$?
- In M_n, are all correctly $\utilde{\Delta}^1_{n+1}$ sets homogeneously Suslin?

[11] Even in this case, there is a unique $\mathcal{U}$-good branch c. For suppose $c \neq c'$ are both $\mathcal{U}$-good. Compare $Q(c,\mathcal{U})$, $Q(c',\mathcal{U})$ and $Q(b,\mathcal{T})$ simultaneously. This comparison succeeds, and all three trees lead to the same final model, with no drops on main branches. A contradiction results as in the proof of Subclaim 1.

3.10. Remark. In M_n, all $(\delta_0^{M_n}+1)$-universally Baire sets are determined, by [11, 6.17]. The Wadge game for comparing two such sets has $(\delta_0^{M_n}+1)$-universally Baire payoff. Therefore all such sets are Wadge comparable, so either all hom sets are $\underset{\sim}{\Pi}^1_n$, or all $\underset{\sim}{\Sigma}^1_n$ sets are hom. (Moreover, let A be the set of towers $\bar{\mu}$ on $M = \mathrm{Hull}_\omega^{M_n}(\emptyset)$ such that $\mathrm{Ult}(M,\bar{\mu})$ is iterable, and let B be complete Π^1_n. Then all hom sets are $\underset{\sim}{\Pi}^1_n$ iff $A \leq_W B$.) To prove that all hom sets are $\underset{\sim}{\Pi}^1_n$, it therefore suffices to prove that all hom co-hom sets are $\underset{\sim}{\Pi}^1_n$.

Also, by the determinacy mentioned above, M_n has well-defined hom thresholds (at any ordinal λ). So a problem related to 3.9 is to determine these. By (relativized versions of) 3.2, combined with [8], certainly the hom threshold of M_n at $\delta_{i+1}^{M_n}$ is $> \delta_i^{M_n}$, but that is all that is known to the author.

Other related questions arise from the proof of 3.2. For instance, in the case of M_1, are conditions (a)-(d) of Claim 1 equivalent to "$\bar{U}$ is wellfounded", or to "$\bar{U}$ is a normal iterate of M"? If so, then certainly all hom sets of M_1 would be $\underset{\sim}{\Pi}^1_1$. In §5 we will discuss a related result. But first, we establish the lemma invoked in the proof of 3.2(b).

§4. Universally Baire sets in M_n. The following lemma has most likely been observed previously by others, but it appears that it's not in print. John Steel provided the proof for $n > 1$. [12] The proof uses the extender algebra and genericity iterations; see [20, §7.2] and 2.1.

4.1. Lemma (Folklore). *In M_n, a set of reals is $\mathrm{Col}(\omega,\delta_0^{M_n})$-universally Baire iff it is correctly $\underset{\sim}{\Delta}^1_{n+1}$.*

Proof of Lemma 4.1. The forward direction is a standard argument; we just sketch it. See [17, 1.2] and [18, 3.0.1] for related arguments.

In M_n, let T, S be the $<_{M_n}$-least (δ_0+1)-absolutely complementing trees whose projections are not correctly $\underset{\sim}{\Delta}^1_{n+1}$, and η an M_n-indiscernible. Let $M = \mathrm{Hull}_\omega^{M_n|\eta}(\emptyset)$. We'll show $p[T]$ is correctly $\Delta^1_{n+1}(M)$ in M_n, a contradiction. Let $\bar{T}, \bar{S}$ be the collapses of T, S. So in M, $\mathrm{Col}(\omega,\delta_0^M)$ forces "$\bar{T},\bar{S}$ project to complements". Let Σ be M's unique iteration strategy.

In V, define complementary sets $A_T, A_S \subseteq \mathbb{R}$ as follows. Let $x \in A_T$ iff there's a tree $\mathcal{U}$ on M via Σ, $i^{\mathcal{U}}$ exists, and $x \in p[i^{\mathcal{U}}(\bar{T})]$. Define A_S likewise. Use genericity iterations to see $A_T \cup A_S = \mathbb{R}$. To see $A_T \cap A_S = \emptyset$: if $y \in A_T \cap A_S$, witnessed by $\mathcal{U},\mathcal{W}$, compare their last models P, Q, and reach a contradiction as in the last paragraph of the proof of [18, 3.0.1] (with $N = M$, $\Psi = P$, $\Gamma = Q$).

Now as in the comments preceding 3.5, A_T and A_S are $\Sigma^1_{n+1}(M)$. And if $x \in A_T \cap M_n$, as witnessed by $\mathcal{U}$, then $x \in p[T]$: argue like in the proof of "(a) $\Rightarrow$ (b)" of Claim 1 of 3.2, with $\mathcal{U}$'s last model replacing $\bar{U}_x$. Likewise with A_S. So in M_n, $p[T]$ is correctly $\Delta^1_{n+1}(M)$, as required.

Now we prove the converse. Consider first M_1. In M_1, if A is correctly $\underset{\sim}{\Delta}^1_2$, then Shoenfield trees for A and its complement witness the (δ_0+1)-universal Baireness of A.

[12] In [15, 4.1], Steel gives another argument which can be used to prove 4.1 for odd $n > 1$. This uses the homogeneous tree construction from [8].

Now consider M_2. If G is M_2-generic for $\mathrm{Col}(\omega,\delta_0^{M_2})$, $M_2[G]$ computes $(\Pi^1_3)^V$ truth with its extender algebra at $\delta_1^{M_2}$, as in 2.1(b). This leads to a tree $T\in M_2$ projecting to $(\Pi^1_3)^V\cap M_2[G]$ as follows. Let $\varphi(u,v)$ be Σ^1_2. Fix $\eta>\delta_1^{M_2}$ such that $M_2|\eta\models\mathsf{ZF}^-$. Let T be the tree building (x,N,π,g), where $\pi:N\to M_2|\eta$ is elementary, g is N-generic for $\mathrm{Col}(\omega,\delta_0^N)$, and $x\in\mathbb{R}^{N[g]}$, and

$$N[g]\models\emptyset\Vdash_{\mathbb{B}^{\delta_1^N}}\varphi(x,x^{\delta_1^N}). \tag{10}$$

(See before Fact 2.1 for the notation.) Using genericity iterations one can show

$$p[T]\cap M_2[G]=\{x\in\mathbb{R}^{M_2[G]}\mid V\models\forall y\varphi(x,y)\}.$$

Now if A in M_2 is (correctly) $\utilde{\Delta}^1_3$, then we get such trees for A and its complement, so A is $(\delta_0^{M_2}+1)$-universally Baire in M_2.

In M_3, one defines a tree T such that in $M_3[G]$, T projects to $(\Sigma^1_4)^V\cap M_3[G]$ (when G is M_3-generic for $\mathrm{Col}(\omega,\delta_0^{M_3})$). Let $\varphi(u,v,w)$ be Σ^1_2 and $\eta>\delta_2^{M_3}$ such that $M_3|\eta\models\mathsf{ZF}^-$. The definition of T is as in the previous case, except that "M_3" replaces "M_2", "$\varphi(u,v,w)$" replaces "$\varphi(u,v)$", and

$$N[g]\models\exists p_1\in\mathbb{B}^{\delta_1^N}\left[p_1\Vdash_{\mathbb{B}^{\delta_1^N}}\left[\emptyset\Vdash_{\mathbb{B}^{\delta_2^N}}\varphi(x,x^{\delta_1^N},x^{\delta_2^N})\right]\right] \tag{11}$$

replaces (10). One can check that this works.[13] □

§5. Towers of measures on M_1. In this section we extract a little more from the proof of 3.2 in the $n=1$ case. Here and in the next section, we use 2.4 implicitly, dispensing with the "$\mathcal{J}_1$"s. We also use some notation from §3.

Let P be a 1-small premouse, either proper class or with $\rho_1^P=\omega$. Let $\mathcal{T}$ be a normal iteration tree on P. Recall that $\mathcal{T}$ is *maximal* iff $\mathcal{T}$ has limit length and $L[M(\mathcal{T})]$ satisfies "$\delta(\mathcal{T})$ is Woodin", or the Q-structure $L_\gamma[M(\mathcal{T})]$ is unsound, or $\mathcal{T}$ has length $\lambda+1$ for some limit λ and $\mathcal{T}\restriction\lambda$ is maximal. In the latter case, note that $i^{\mathcal{T}}$ exists and there is no normal extension of $\mathcal{T}$.

5.1. Theorem. *Suppose M_1 satisfies "C is a countable set of measures", μ is a wellfounded tower on M_1 and $\mu_n\in C$ for all n (possibly $\mu\notin M_1$). Then*

(a) $\mathrm{Ult}(M_1,\mu)$ *is fully iterable.*
(b) *The comparison between* $\mathrm{Ult}(M_1,\mu)$ *and* M_1 *is non-maximal.*
(c) *If $\mu\in M_1[G]$ for some M_1-generic G, then $M_1[G]$ satisfies "there is a countable non-maximal normal iteration tree $\mathcal{T}$ on $L[\mathbb{E}]$ and an embedding ψ from* $\mathrm{Ult}(L[\mathbb{E}],\mu)$ *to $\mathcal{T}$'s last model".*

Proof. Because the statement of the theorem isn't first order over M_1, we can't reduce to a minimal counter-example. So let $M=\mathrm{Hull}_\omega^{M_1|\eta}(\{C\})$ where η is an M_1-indiscernible. Let $\pi:M\to M_1|\eta$ be the uncollapse and $\pi``\bar\mu=\mu$.

Claim 1. *$\bar U=\mathrm{Ult}(M,\bar\mu)$ is Π^1_2-iterable.*

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[13]This method also provides a proof of 2.1(a) that avoids using the stationary tower. For example, to prove that $M_3[G]$ is Σ^1_4-correct, suppose $b\in M_3[G]$ and $\exists y\forall z\varphi(b,y,z)$. Let $(b,N,\pi,g)\in M_3[G]$ be a branch through T. Let $p_1\in N[g]$ witness (11) and in $M_3[G]$, let h be $N[g]$-generic for $\mathbb{B}^{\delta_1^N}$ with $p_1\in h$. Then $y=(x^{\delta_1^N})^h\in M_3[G]$ and $\forall z\varphi(b,y,z)$.

Proof. The Shift Lemma gives an elementary

$$\pi' : \mathrm{Ult}(M, \bar{\mu}) \to \mathrm{Ult}(M_1|\eta, \mu) = \mathrm{Ult}(M_1, \mu)|i_\mu(\eta).$$

Now M_1 satisfies "In $V^{\mathrm{Col}(\omega,\eta)}$, every countable elementary substructure of $L[\mathbb{E}]|\eta$ is Π^1_2-iterable". Therefore the same thing holds in $\mathrm{Ult}(M_1, \mu)$ about $i_\mu(\eta)$. But if G is generic for $\mathrm{Col}(\omega, i_\mu(\eta))$, then $\mathrm{Ult}(M_1, \mu)[G]$ is Σ^1_2-correct, so $\bar{U}$ is indeed Π^1_2-iterable. □(Claim 1)

Let Σ be any ω_1-iteration strategy for M. [14] By Claim 1 and the proof of the $n = 1$ case of 3.2, there's a successful comparison $(\mathcal{T}, \mathcal{U})$ of M vs $\bar{U}$, with $\mathcal{T}$ via Σ, producing a common final model Q, and $i^{\mathcal{T}}$ and $i^{\mathcal{U}}$ exist.

Claim 2. $i^{\mathcal{T}} = i^{\mathcal{U}} \circ i_{\bar{\mu}}$.

Proof. Let $\pi(A) = C$ and $A^Q = i^{\mathcal{T}}(A)$. It suffices to see $A^Q = i^{\mathcal{U}}(i_{\bar{\mu}}(A))$.

Let $T = \mathrm{Th}^M_\omega(\{A\})$. We claim that [15] there is no $B <_Q A^Q$ such that $\mathrm{Th}^Q_\omega(\{B\}) = T$. Otherwise let $\mathcal{V} = \mathcal{T}_{\{B\}}$ be the finite support tree, with last model R and $\varrho : R \to Q$ given by 2.9. Let $\varrho(B') = B$. So there is a (unique) elementary $\pi' : M \to R$ such that $\pi'(A) = B'$. But by 2.9, $\varrho(i^{\mathcal{V}}(A)) = A^Q$, so $\pi'(A) <_R i^{\mathcal{V}}(A)$. But a Dodd-Jensen argument shows this is impossible. (Because $\mathcal{V}$ is finite, we just need M sufficiently iterable for this - it's irrelevant what properties Σ has.) [16]

So $i^{\mathcal{U}}(i_{\bar{\mu}}(A)) \geq_Q A^Q$. Assume it's "$>_Q$". Let $\mathcal{V}_1$ on $\bar{U}$ be the finite support tree $\mathcal{U}_{\{A^Q\}}$. Then $\mathcal{V}_1 \in \bar{U}$; let n and $\mathcal{V}_1'$ be such that $\mathcal{V}_1 = \bar{\jmath}_{n,\infty}(\mathcal{V}_1')$. By [14, 4.8], there is [17] a finite normal tree $\mathcal{S}$ on M, with final model $M^{\mathcal{S}}_k = \mathrm{Ult}(M, \bar{\mu}_n)$, and no dropping on the main branch, and such that $i^{\mathcal{S}} = i_{\bar{\mu}_n}$. Let $\mathcal{V} = \mathcal{S}\,^\frown \mathcal{V}_1'$. So $\mathcal{V}$ is a stack of 2 finite normal trees, with base model M and last model $M^{\mathcal{V}}_k$, and there is a $\pi : M \to M^{\mathcal{V}}_k$ with $\pi(A) <_{M^{\mathcal{V}}_k} i^{\mathcal{V}}(A)$. Again a Dodd-Jensen argument provides a contradiction. □(Claim 2)

We now assume that Σ is the strategy given by lifting to M_1 via π. (For this reason, we didn't assume the weak Dodd-Jensen property for Σ earlier.) Now by 3.1 and Claim 2, $\mathrm{Ult}(M_1, \mu)$ embeds into the final model of $\pi\mathcal{T}$. Also since

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[14] See footnote 16.

[15] A direct argument shows there is no $A_1 <_M A$ such that $T = \mathrm{Th}^M_\omega(\{A_1\})$: Since $M = \mathrm{Def}^M_\omega(\{A\})$, the existence of such an A_1 would be coded into T, so we could define an infinite decreasing sequence $A_{n+1} <_M A_n$. Nor is there any $A_1 > A$ definable from A and with the same theory. For otherwise let $A_0 = A$, and let A_{n+1} be defined from A_n just as A_1 is from A_0. Let γ_n be the least ordinal from which A_n is definable. Then $\gamma_{n+1} < \gamma_n$ for all n.

[16] This argument shows M has a unique ω_1-iteration strategy. Given a tree $\mathcal{T}$ with "strategic" branches $b \neq c$, $\mathcal{T}$ must be maximal. Use 3.8 to show that $i^{\mathcal{T}}_b \neq i^{\mathcal{T}}_c$ and then use the proof just given for a contradiction. However, this doesn't seem to rule out the possibility of having multiple embeddings from M to some iterate, if they're not branch embeddings.

[17] For our use of $\mathcal{S}$, it suffices to just show that M satisfies "For every measure λ, there's a finite normal tree $\mathcal{S}$ on me, with no dropping on the main branch, and $l \in \mathrm{OR}^{<\omega}$, such that λ is the measure derived from $i^{\mathcal{S}}$ and l". (Given this, if $\mathcal{S}$ has these properties with respect to $\bar{\mu}_n$, the rest of the proof goes through with slight adjustment.) If M thinks otherwise, then M_1 and $N = \mathrm{Hull}^{M_1}_\omega(\emptyset)$ agree. But given a measure $\lambda \in N$ generated by $a \in (\mathrm{OR}^N)^{<\omega}$, earlier arguments show $\mathrm{Ult}(N, \lambda)$ is iterable. Let $(\mathcal{T}, \mathcal{U})$ be the successful comparison of N vs $\mathrm{Ult}(N, \lambda)$, producing a common model Q. Then $i^{\mathcal{T}}$ and $i^{\mathcal{U}}$ exist, and the embeddings commute. Let $\mathcal{S}$ be the finite support tree $\mathcal{T}_{i^{\mathcal{U}}(a)}$ and $l = \bar{\varrho}^{-1}(i^{\mathcal{U}}(a))$, where $\bar{\varrho}$ is as in 2.9. Then $\mathcal{S}, l$ witness the truth of the statement in N, a contradiction.

$\pi : M \to M_1$ is bounded in δ^{M_1}, $\pi\mathcal{T}$ is a non-maximal tree. If $\mu \in M_1[G]$, then everything has taken place there. This proves (a) and (c).

By (a), we have a successful comparison $(\mathcal{T},\mathcal{U})$ of M_1 vs $\mathrm{Ult}(M_1,\mu)$. We now show this is non-maximal. Let Q be the final model and b, c be the main branches of $\mathcal{T},\mathcal{U}$ respectively. Let $\nu(\mu)$ be the sup of generators of μ. Then $\nu(\mu) < \delta^{M_1}$ since C is countable in M_1. Now assume the trees are maximal. Then there's a least $\alpha \in c$ such that $i^{\mathcal{U}}_{0,\alpha}$"$\nu(\mu) \subseteq \mathrm{crit}(i^{\mathcal{U}}_{\alpha,c})$. Let $\kappa = \mathrm{crit}(i^{\mathcal{U}}_{\alpha,c})$. We will use the hull property argument (cf [16, §4]). Let Γ be the class of uncountable cardinals. So Γ is fixed point-wise by $i^{\mathcal{T}}$, $i^{\mathcal{U}}$, and i_μ. Since $\delta^{M_1} \subseteq \mathrm{Def}^{M_1}_\omega(\Gamma)$, we have

$$M^{\mathcal{U}}_\alpha|\delta^{M^{\mathcal{U}}_\alpha} \subseteq \mathrm{Def}^{M^{\mathcal{U}}_\alpha}_\omega(\Gamma\cup\kappa) \cong \mathrm{Def}^{Q}_\omega(\Gamma\cup\kappa), \tag{12}$$

with $i^{\mathcal{U}}_{\alpha,c}$ exhibiting the isomorphism. So $\mathcal{P}(\kappa)^Q \subseteq \mathrm{Hull}^Q_\omega(\Gamma\cup\kappa)$, so as in [16, 4.3], there's no E used along b overlapping κ. So let $\beta \in b$ be least such that $\kappa \leq \mathrm{crit}(i^{\mathcal{T}}_{\beta,b})$. Then (12) holds with "$M^{\mathcal{T}}_\gamma$" replacing "$M^{\mathcal{U}}_\alpha$", and this leads to $M^{\mathcal{U}}_\alpha = M^{\mathcal{T}}_\gamma = Q$, contradicting the choice of α. □(Theorem 5.1)

The assumption of countability of the set of measures, or something like it, is necessary for the preceding theorem, as the next example shows. For a premouse R with a least Woodin δ, let $\mathbb{P}^R$ be the δ-generated extender algebra of R at δ.

5.2. *Example.* There is an M_1-generic G for $\mathbb{P}^{M_1}$ and a $\mu \in M_1[G]$, $\mu \subseteq M_1$, such that μ is a tower on M_1 and $\mathrm{Ult}(M_1,\mu)$ is wellfounded, but not iterable. Moreover, μ may be taken to form an extender whose generators have ordertype ω, and such that $(M_1||\mathrm{lh}(\mu),\mu)$ is a sound mouse.

To see this, let E be the first total type 3 extender on $\mathbb{E}^{M_1}$. Let N_1 be the final model of the linear iteration $\mathcal{T}_1$ on M_1 given by hitting E and its images δ^{M_1} times. Let $\mathcal{T}_2$ be the genericity iteration on N_1 (which is above $i^{\mathcal{T}_1}(\mathrm{crit}(E)+1)$), making $M_1|\delta^{M_1} \frown \langle 0,0,0,\dots\rangle$ generic for $\mathbb{P}^P$ where P is the last model of $\mathcal{T}_2$. It is a standard fact that $\mathcal{T}_2 = \mathcal{T}_2' \frown b$ is a maximal iteration, with $\mathcal{T}_2' \in M_1$ and $b \notin M_1$, and the universe of $P[M_1|\delta^{M_1}]$ is that of M_1.

Now $\mathcal{T} = \mathcal{T}_1 \frown \mathcal{T}_2' \in M_1$ is a normal tree on M_1, $M(\mathcal{T}) \in M_1$, and $P = L(M(\mathcal{T}))$. Clearly $\nu(E) = (\mathrm{crit}(E)^{++})^P$, $\mathrm{lh}(E) = (\mathrm{crit}(E)^{+++})^P$, $(P|\mathrm{lh}(E),E) = M_1|\mathrm{lh}(E)$ is a sound premouse, and in P, the least total type 3 extender on $\mathbb{E}$ has critical point $> \mathrm{crit}(E)$. Moreover, $\mathrm{Ult}(P,E)$ is wellfounded since $P \subseteq M_1$. These and other facts (which we'll add later) are forced by some $i_{M_1,P}(p)$ in $\mathbb{P}^P$. Let $i_{M_1,P}(\dot{F})$ be a name for E.

Now let H be M_1-generic over $\mathbb{P}^{M_1}$ with $p \in H$, and let $F = \dot{F}^H$. So $\mathrm{Ult}(M_1,F)$ is wellfounded. We claim that $\mathrm{Ult}(M_1,F)$ is not iterable. Otherwise, let $(\mathcal{T},\mathcal{U})$ be the successful comparison of M_1 vs $\mathrm{Ult}(M_1,F)$, with final model Q; $i^{\mathcal{T}}$ and $i^{\mathcal{U}}$ exist. Since E cohered P's sequence, we know F coheres M_1's sequence below $\mathrm{lh}(F)$, so the least difference between M_1 and $\mathrm{Ult}(M_1,F)$ is $\geq \mathrm{lh}(F)$. But $\mathrm{lh}(F) = (\mathrm{crit}(F)^{+++})^{M_1}$ is a cutpoint of M_1 (the first extender overlapping $\mathrm{lh}(F)$ would be total type 3, but $\mathrm{crit}(F) < \mathrm{crit}(E)$). So $\mathrm{crit}(i^{\mathcal{T}}) > \mathrm{lh}(F)$. But $M_1 = \mathrm{Hull}^{M_1}_\omega(\mathrm{Card})$, and all cardinals are fixed by $i^{\mathcal{T}}$, $i^{\mathcal{U}}$ and $i^{M_1}_F$. It follows that $i^{\mathcal{T}} = i^{\mathcal{U}} \circ i^{M_1}_F$, so $\mathrm{crit}(i^{\mathcal{T}}) \leq \mathrm{crit}(F)$, a contradiction.

Now $(M_1|\mathrm{lh}(F), F)$ is a sound premouse, since this was true of $(P|\mathrm{lh}(E), E)$. It is iterable because $P[M_1|\delta^{M_1}]$ has an embedding $M_1|\mathrm{lh}(E) \to i^{\mathcal{T}}(M_1|\mathrm{lh}(E))$, so $M_1[H]$ has an embedding $(M_1|\mathrm{lh}(F), F) \to M_1|\mathrm{lh}(E)$. [18]

§6. Homogeneously Suslin sets in $0^\P$ and M_ω. In this section we turn to some mice below, and some above, the M_n's. Recall that $0^\P$ is the least active mouse N such that $N|\mathrm{crit}(F^N)$ satisfies "there is a strong cardinal". For mice modelling ZFC in the region of $0^\P$ or below, we get an exact characterization of homness.

6.1. THEOREM. *Let $N \models$ ZFC be an $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse satisfying "For all measurables κ and all $\mu < \kappa$, μ is not strong to κ" (call this statement ψ). Then in N, all homogeneously Suslin sets are $\utilde{\Pi}^1_1$. Therefore $0^\P$ satisfies "the homogeneously Suslin sets are precisely the $\utilde{\Pi}^1_1$ sets".*

PROOF. [19] This is essentially a corollary of the proof of 3.2 and the fact, probably due to Jensen (cf [13, §0]) that below $0^\P$, every mouse is an iterate of its core. We use notation like that in 3.1, 3.2 and their proofs, but "N" here replaces "M_n" in 3.2. Let T be the $<_N$-least contrary homogeneous tree and ν be its $<_N$-least homogeneity system. Here we don't have indiscernibles, so let $\eta = (\theta^+)^N$ where θ is sufficiently large and reflective. (E.g. $T, \nu \in N|\theta$ and $N|\theta \preccurlyeq_5 N$.) In particular, θ is a limit cardinal of N, and $N|\theta$ is correct about "hom", etc. For all $z \in \mathbb{R}^N$, $\mathrm{Ult}(N, \nu_z)$ is wellfounded iff $\mathrm{Ult}(N|\theta, \nu_z)$ is.

Now let $M = \mathrm{Hull}_\omega^{N|\eta}(\emptyset)$ and $\pi : M \to N|\eta$ be elementary; note that $M \models \psi$ and $T, \nu \in \mathrm{rg}(\pi)$. Let Σ be M's unique strategy (M could have proper segments with Woodin cardinals); this is given by lifting to $N|\eta$. Let $\bar{\mu} \in N$ by a tower on M, and $\mu = \pi``\bar{\mu}$. By 3.1 and the proof of Claim 1 of 3.2, $\mathrm{Ult}(N, \mu)$ is wellfounded iff $\bar{U} = U^M_{\bar{\mu}}$ is iterable in V. We will show that the statement "$U^M_{\bar{\mu}}$ is iterable" is $\Pi^1_1(M)$, completing the proof.

For any $\bar{\mu}$ (maybe $\bar{\mu} \notin N$) such that $\bar{U} = U^M_{\bar{\mu}}$ is iterable, let $(\mathcal{T}, \mathcal{U})$ be the comparison of M vs $\bar{U}$. This produces a common last model Q, with non-dropping main branches b, c, respectively. We claim $\mathcal{U}$ is trivial; i.e. $\bar{U} = Q$. (This is essentially Jensen's fact, since the "core" of $\bar{U}$ is M. However, our mice aren't literally below $0^\P$.)

Suppose $\mathcal{U}$ is non-trivial. Let $\kappa = \mathrm{crit}(i^{\mathcal{U}})$. Then κ is measurable in $\bar{U}$. Since $\bar{U} \models \psi$, no $\mu < \kappa$ is $< \kappa$-strong in $\bar{U}$, or therefore in Q. So for all α, $(*)$ if $\mathrm{crit}(E^{\mathcal{T}}_\alpha) < \kappa$ then $\mathrm{lh}(E^{\mathcal{T}}_\alpha) < \kappa$. Now we get a contradiction via the definability and hull properties: There is $E^{\mathcal{T}}_\alpha$ used along b of which κ is a generator, since $\kappa \notin \mathrm{Def}^Q_\omega(\kappa)$. By $(*)$, $\mathrm{crit}(E^{\mathcal{T}}_\alpha) = \kappa$. So if $\gamma = \mathcal{T}\text{-pred}(\alpha + 1)$ then

$$\mathrm{rg}(i^{\mathcal{T}}_{\gamma,b}) = \mathrm{Def}^Q_\omega(\kappa) \subseteq \mathrm{rg}(i^{\mathcal{U}}),$$

and $\mathcal{P}(\kappa)^{\bar{U}} = \mathcal{P}(\kappa)^Q = \mathcal{P}(\kappa)^{M^{\mathcal{T}}_\gamma}$. This leads to $E^{\mathcal{T}}_\alpha$ being compatible with the first extender used along c, a contradiction.

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[18]In fact, $i^{\mathcal{T}}(M_1|\mathrm{lh}(E))$ is a linear iterate of $M_1|\mathrm{lh}(E)$ by iterating its top extender. For if $\kappa = \mathrm{crit}(E)$, then $(\kappa^+)^{M_1}$ is the largest cardinal of $M_1|\nu(E)$, and $\nu(E)$ has cofinality ω in M_1. Therefore this iteration uses the same functions to form ultrapowers as those used in forming $i^{\mathcal{T}}(M_1|\mathrm{lh}(E))$.

[19]Thanks to the referee for questions leading to corrections to our original argument.

Now if $\mathcal{T}'$ is any normal tree on M with a non-dropping main branch, then for every limit $\lambda < \mathrm{lh}(\mathcal{T}')$, $M(\mathcal{T}' \upharpoonright \lambda)$ is the Q-structure for itself, since $M \models \psi$ and $\delta(\mathcal{T}' \upharpoonright \lambda)$ is a cardinal of the last model of $\mathcal{T}'$. Therefore, any such tree is via Σ. But $\bar{U}$ is iterable iff $\bar{U}$ is a non-dropping Σ-iterate of M, and the latter condition is therefore $\Pi^1_1(M)$. [20] □

John Steel and Hugh Woodin observed that the arguments from earlier sections adapt to M_ω. The author then generalized this to certain tame mice. Recall that a premouse N is *tame* iff for all active $P \trianglelefteq N$, if $\mathrm{crit}(F^P) \leq \delta < \mathrm{OR}^P$ then $P \not\models$ "δ is Woodin". We work with tame mice $N \models \mathsf{ZFC}$ + "I have a sufficiently absolute iteration strategy for $L[\mathbb{E}]|\omega_1$ which lifts well to segments of $L[\mathbb{E}]$".

6.2. Lemma. *Let N be a tame premouse satisfying* ZFC*, "my levels satisfy condensation" and "λ is a limit cardinal". Let $S \in N$ be a tree on $\omega \times \lambda$. Suppose that whenever G is $<\lambda$-generic over N,*

$$N[G] \models \text{"}p[S] \text{ codes an } (\omega,\omega_1)\text{-iteration strategy } \Sigma_S \text{ for } N|\omega_1^N\text{"}.$$

Suppose that for all $\alpha < \omega_1^N$ and $\theta \in \mathrm{OR}^N$ such that $N|\alpha = \mathrm{Hull}_\omega^{N|(\theta^+)^N}(\emptyset)$, if $\pi : N|\alpha \to N|(\theta^+)^N$ is elementary, and if G is $<\lambda$-generic over N, then

$$N[G] \models \text{"For any } \mathcal{T} \text{ on } N|\alpha \text{ via } \Sigma_S, \text{ the models of } \pi\mathcal{T} \text{ are wellfounded"}.$$

Then in N, every hom set is λ-universally Baire. [21]

Proof. We start by establishing some basic claims. The first is standard.

Claim 1. *Let G be $<\lambda$-generic over N. Then $N[G]$ satisfies "$N|\omega_1^N$ is λ-iterable, via a strategy extending Σ_S".*

Proof. $N[G]$ uniformly computes $\Sigma_S^{N[G][H]} \upharpoonright N[G]$, for any $<\lambda$-generic H. □

Claim 2. *Let $\gamma \in \mathrm{OR}^N$, $H \in \mathrm{HC}^N$ and $\pi \in N$ be such that $\pi : H \to N|\gamma$ is elementary. Let $\bar{\mu} \in N$ be a tower on H, such that $\mu = \pi``\bar{\mu}$ is a wellfounded tower on N. Then there is $\alpha < \omega_1^N$ and an elementary $\tau : \mathrm{Ult}(H,\bar{\mu}) \to N|\alpha$ with $\tau \in N$.*

Proof. Suppose not. Let $(\gamma,\pi,\bar{\mu})$ be the $<_N$-least counterexample. Let $N|\theta \preccurlyeq_5 N$ and $\eta = (\theta^+)^N$. Let $M = \mathrm{Hull}_\omega^{N|\eta}(\emptyset)$ and $\pi' : M \to N|\eta$ be elementary. So $(\gamma,\pi,\bar{\mu}) \in \mathrm{rg}(\pi')$, and by 2.4 and Σ_1-condensation, $\mathcal{J}_1(M) \triangleleft N|\omega_1^N$.

As in the proof of 3.2, there's an elementary $\sigma : \mathrm{Ult}(H,\bar{\mu}) \to N|\gamma$ with $\sigma \in N|\eta$, and we may take $\sigma \in \mathrm{rg}(\pi')$. Let $\pi'(\bar{\gamma}) = \gamma$. Since π' fixes $H, \bar{\mu}$, in M there's

$$\tau : \mathrm{Ult}(H,\bar{\mu}) \to M|\bar{\gamma} \triangleleft M \triangleleft N|\omega_1^N,$$

with τ elementary, as required. □

Now let θ, η, M and π be defined as in 6.1. The $<_N$-least λ, S and homogeneity system contrary to the theorem will be in $\mathrm{rg}(\pi)$. So to prove 3.2, it suffices to find λ-absolutely complementing trees W, I (well, ill) in N, such that N satisfies "$p[W]$ is the set of towers $\bar{\mu}$ on M such that $\mathrm{Ult}(N,\pi``\bar{\mu})$ is wellfounded".

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[20] One clause of the condition is the assertion "$\bar{U}$ is wellfounded", so it's not $\Sigma^1_1(M)$.

[21] By "via Σ_S" we mean via the canonical strategy induced by Σ_S on $N|\alpha$. By 2.4, $\mathcal{J}_1(N|\alpha)$ projects to ω, so Σ_S acts directly on $\mathcal{J}_1(N|\alpha)$, which amounts to acting directly on $N|\alpha$ anyway.

CLAIM 3. *Let $\mu \in N$ be a tower on M. The following are equivalent in N:*

(i) $\mathrm{Ult}(N, \mu)$ *is wellfounded, where* $\mu = \pi``\bar{\mu}$*;*
(ii) $\mathrm{Ult}(M, \bar{\mu})$ *is* λ*-iterable;*
(iii) *there is a successful comparison* $(\mathcal{T}, \mathcal{U})$ *of* M *vs* $\mathrm{Ult}(M, \bar{\mu})$*, with* $\mathcal{T}$ *via* Σ_S*.*

PROOF. For "(iii) $\Rightarrow$ (i)": Our hypotheses ensure $\pi\mathcal{T}$ has wellfounded models. So by 3.1, $\mathrm{Ult}(N|\eta, \mu)$ is wellfounded, so $\mathrm{Ult}(N, \mu)$ is also, by choice of θ. For "(i) $\Rightarrow$ (ii)" and "(ii) $\Rightarrow$ (iii)", use Claims 2 and 1. □

The main idea for the remainder of the proof is similar to that used in [1, 2.1,2.8]; [1, 2.1] is attributed there to folklore.

Motivated by Claim 3, we define W such that whenever G is $<\lambda$-generic over N, $N[G]$ satisfies "For all towers $\bar{\mu}$ on M, $\bar{\mu} \in p[W]$ iff condition (iii) of Claim 3 holds". One uses S to define W with this property. We omit the details, but they're essentially contained in the definition of I below.

We will define I to be a tree of attempts to build (a) a tower $\bar{\mu}$, and (b) a proof that there's no comparison as in (iii) of Claim 3. To see what (b) should consist of, assume that $(\mathcal{T}, \mathcal{U})$ is a successful comparison of M vs $\bar{U} = \mathrm{Ult}(M, \bar{\mu})$. This gives a common final model Q, and $i^{\mathcal{T}}, i^{\mathcal{U}}$ exist. We will analyze branch choices in $\mathcal{U}$. We take $\mathcal{T}$ and $\mathcal{U}$ to be padded as usual. If $\gamma < \mathrm{lh}(\mathcal{U})$ is a limit ordinal, we'll say it's an *actual limit stage* of $\mathcal{U}$ if $\mathcal{U}$ has non-padded stages cofinally in γ. Let $\gamma < \mathrm{lh}(\mathcal{U})$ be an actual limit stage of $\mathcal{U}$ and $\delta = \delta(\mathcal{U} \upharpoonright \gamma)$. We now describe how to identify $[0, \gamma]_{\mathcal{U}}$ from $\mathcal{T} \upharpoonright \gamma + 1$ and $\mathcal{U} \upharpoonright \gamma$. There are two cases.

Case 1. *δ is Woodin in $M^{\mathcal{T}}_\gamma$.*

Then $\mathcal{U} = (\mathcal{U} \upharpoonright \gamma + 1) \frown \mathcal{U}'$, with $\mathcal{U}'$ a normal tree on $M^{\mathcal{U}}_\gamma$, above δ. (So both $\mathcal{U} \upharpoonright \gamma + 1$ and $\mathcal{U}'$ have non-dropping main branches.) If $Q = M^{\mathcal{U}}_\gamma$ this is immediate, and otherwise it follows from tameness: if $\gamma' \geq \gamma$ and $E^{\mathcal{U}}_{\gamma'} \neq \emptyset$ then δ is Woodin in $M^{\mathcal{T}}_{\gamma'}|\mathrm{lh}(E^{\mathcal{U}}_{\gamma'}) = M^{\mathcal{U}}_{\gamma'}|\mathrm{lh}(E^{\mathcal{U}}_{\gamma'})$, so $\mathrm{crit}(E^{\mathcal{U}}_{\gamma'}) \geq \delta$.

An analogous claim applies to $\mathcal{T}$. Note δ is Woodin in Q also.

Now, whether or not γ is an actual limit stage of $\mathcal{T}$, let $\alpha < \gamma$ be least such that $\delta \in \mathrm{rg}(i^{\mathcal{T}}_{\alpha,\gamma})$, and let $i^{\mathcal{T}}_{\alpha,\gamma}(\bar{\delta}) = \delta$. If $\alpha = 0$ let $\nu = 0$. Otherwise $\alpha = \beta + 1$; then let $\nu = \nu(E^{\mathcal{T}}_\beta)$. Either way ν is least such that $M^{\mathcal{T}}_\alpha = \mathrm{Hull}^{M^{\mathcal{T}}_\alpha}_\omega(\nu)$.

Let $H = \delta \cap \mathrm{rg}(i^{\mathcal{T}}_{\alpha,\gamma})$. By the previous three paragraphs, $H = \delta \cap \mathrm{Def}^R_\omega(\nu)$, where R is any of the models $M^{\mathcal{T}}_\gamma$, Q or $M^{\mathcal{U}}_\gamma$. Note $\nu < \bar{\delta}$ by tameness, and $i^{\mathcal{T}}_{\alpha,\gamma}$ is continuous at $\bar{\delta}$, since $\bar{\delta}$ is regular in $M^{\mathcal{T}}_\alpha$ and $M^{\mathcal{T}}_\alpha \models \mathsf{ZF}^-$. Therefore H is cofinal in δ.

Now if $\beta \in [0, \gamma]_{\mathcal{U}}$ and $\mathrm{crit}(i^{\mathcal{U}}_{\beta,\gamma}) \geq \nu$ then $H \subseteq \mathrm{rg}(i^{\mathcal{U}}_{\beta,\gamma})$. By 3.8 then, $[0, \gamma]_{\mathcal{U}}$ is the unique $\mathcal{U} \upharpoonright \gamma$-cofinal branch b such that for some $\beta \in b$, $H \subseteq \mathrm{rg}(i^{\mathcal{U}}_{\beta,b})$.

Note also that γ is an actual limit stage of $\mathcal{T}$, since otherwise $\delta \subseteq H$, but this is impossible as $\delta \not\subseteq \mathrm{rg}(i^{\mathcal{U}}_{\beta,\gamma})$ for any $\beta <_{\mathcal{U}} \gamma$.

Case 2. *δ is not Woodin in $M^{\mathcal{T}}_\gamma$.*

Let $M^{\mathcal{T}}_\gamma|\xi \lhd M^{\mathcal{T}}_\gamma$ be the Q-structure for $\delta(\mathcal{U})$. Then $M^{\mathcal{T}}_\gamma|\xi \lhd M^{\mathcal{U}}_\gamma$. (Otherwise, use tameness as in Case 1 to show that $(\gamma, Q]_{\mathcal{T}}$ drops, a contradiction.) So $[0, \gamma]_{\mathcal{U}}$ is the unique branch b through $\mathcal{U} \upharpoonright \gamma$ with $M^{\mathcal{T}}_\gamma|\xi \trianglelefteq M^{\mathcal{U}}_b$. This completes Case 2.

Given a putative partial comparison $(\mathcal{T},\mathcal{U})$ of M vs $\bar{U}$ in $N[G]$, we say $(\mathcal{T},\mathcal{U})$ is *via* Σ_S iff $\mathcal{T}$ is via Σ_S, and $\mathcal{U}$ is formed according to the prescription given by Cases 1 and 2, including that for any $\gamma < \mathrm{lh}(\mathcal{T},\mathcal{U})$ which is an actual limit stage of $\mathcal{U}$, if $[0,\gamma]_{\mathcal{T}}$ drops then $\delta(\mathcal{U}\restriction\gamma)$ is not Woodin in $M^{\mathcal{T}}_\gamma$. [22]

Now we define I. It is the tree on $\omega\times\eta$, building $(\bar{\mu},(P,\sigma,g,\mathcal{T},\mathcal{U}))$, such that:

- $\bar{\mu}$ is a tower on M,
- $\sigma : P \to N|\eta$ is elementary; let $\lambda^P, S^P = \sigma^{-1}(\lambda, S)$,
- g is $<\lambda^P$-generic over P and $\bar{\mu} \in P[g]$,
- $P[g]$ satisfies "$(\mathcal{T},\mathcal{U})$ is a countable putative partial comparison of M vs $\bar{U}$, via Σ_{S^P}, with no proper extension via Σ_{S^P}."

We claim that W and I are λ-absolutely complementing. Let us first verify that $p[W]\cap p[I] = \emptyset$; it suffices to do this in N. If $\bar{\mu} \in p[I]\cap N$, as witnessed by $P,\sigma,g,\mathcal{T},\mathcal{U}$, then $(\mathcal{T},\mathcal{U})$ is via Σ_S, since σ"$S^P \subseteq S$. Moreover, $b = \Sigma_S(\mathcal{T}) \in P$, so P has the tree R of attempts to build a $\mathcal{U}$-cofinal branch c such that $(\mathcal{T}\,^\frown b,\mathcal{U}\,^\frown c)$ is via Σ_S. (R refers to b, not to S.) So R is wellfounded in P, and so also in N. So N has no such branch c. But a successful comparison via Σ_S would have to extend $(\mathcal{T},\mathcal{U})$, by the analysis above. So $\bar{\mu}\notin p[W]$, as required.

Now let G be $<\lambda$-generic and $\bar{\mu}\in N[G]$; suppose $\bar{\mu}\notin p[W]\cup p[I]$. Then $N[G]$ computes a partial comparison $(\mathcal{T},\mathcal{U})$ of M vs $\bar{U}$, via Σ_S, of length $\omega_1^{N[G]} < \lambda$. Let $b = \Sigma(\mathcal{T})$; $b \in N[G]$ as in Claim 1. By standard arguments, $\omega_1^{N[G]}$ is the critical point of an extender on the sequence of $M^{\mathcal{T}}_b$, so by tameness, $\omega_1^{N[G]}$ is not Woodin in $M^{\mathcal{T}}_b$. Let $M^{\mathcal{T}}_b|\xi \in N[G]$ be the Q-structure. If there is a $\mathcal{U}$-cofinal branch c such that $M^{\mathcal{T}}_b|\xi \lhd M^{\mathcal{U}}_c$, then $c\in N[G]$ by uniqueness and the homogeneity of $\mathrm{Col}(\omega,\omega_1^{N[G]})$. But then $(\mathcal{T}\,^\frown b,\mathcal{U}\,^\frown c)$ contradicts comparison termination in $N[G]$. So there is no such c, which implies $\bar{\mu}\in p[I]$, a contradiction. □

We can apply 6.2 to tame mice N with a limit cardinal λ, such that all $<\lambda$-generic extensions $N[G]$ correctly iterate $N|\omega_1^N$ by choosing branches guided by Q-structures built by an $L[\mathbb{E}]$ construction above G, below λ. For example:

6.3. Lemma. *Let N be (a) M_ω, (b) the sharp for a proper class of Woodins, or (c) the least non-tame mouse. In cases (a), (b) let λ be the sup of the Woodins in N. In case (c) let $\lambda = \mathrm{crit}(F^N)$. Then N,λ meet the requirements of 6.2.*

Proof. The method is standard (cf. the proof of "(b) ⇒ (c)" of [18, 5.1]), but (c) involves a slight variation, so we just sketch this case.

Let Γ be V's iteration strategy for $N|\omega_1^N$. Let $\kappa = \mathrm{crit}(F^N)$. N can compute $\Gamma\restriction V^N_\kappa$, as follows. Let $\mathcal{T}\in N$ be a normal tree on $N|\omega_1^N$, of limit length $<\kappa$. We inductively assume $\mathcal{T}$ is via Γ, and compute $b = \Gamma(\mathcal{T})$. Let $\langle N_\alpha\rangle$ be the models of the maximal fully backgrounded $L[\mathbb{E}]$ construction, as computed in $N|\kappa$, starting with $N_0 = M(\mathcal{T})$. Then there is some N_α such that the $\delta(\mathcal{T})$-core [23] of N_α is a Q-structure Q for $M(\mathcal{T})$, and moreover, if α is least such, then $Q \trianglelefteq M^{\mathcal{T}}_b$, which uniquely identifies b. Here is an outline: Otherwise, let $N^* = i_{F^N}(N_\kappa)|\nu(F^N)$, and attempt to define a squashed premouse $R^{\mathrm{sq}} = (N^*, F^N\restriction N^*)$. Then [9, §11]

[22]The arguments given show that if there's no successful comparison between M and $\bar{U}$, for any α there's still at most one partial comparison $(\mathcal{T},\mathcal{U})$ of $(M,\bar{U})$ via Σ_S of length α.

[23]I.e. $\mathrm{Hull}^{N'}_{n+1}(\delta(\mathcal{T})\cup p^{N'}_{n+1}\cup u^{N'}_n)$, where $N' = \mathfrak{C}_n(N_\alpha)$, and $\rho^{N'}_{n+1}\le\delta(\mathcal{T}) < \rho^{N'}_n$.

shows that R is a premouse, and that $\nu(F^N)$ is Woodin in R, so R is non-tame. A slight variation of [9, §12] shows R is iterable. Also R satisfies "$\delta(\mathcal{T})$ is Woodin" and $M(\mathcal{T}) \trianglelefteq R$. Comparing R with $M_b^{\mathcal{T}}$ leads to a contradiction. [24]

The foregoing generalizes to $N[G]$ if G is N-generic for some $\mathbb{P} \in N|\kappa$, using $L[\mathbb{E}]$ constructions with critical points $> \text{rank}(G)$.

One now defines a tree S on $\omega\times\kappa$ in the usual way. (Have S build $((\mathcal{T}, b), P, g)$, where $\mathcal{T}$ is an iteration tree on $N|\omega_1^N$, $P \preccurlyeq N|\kappa$, g is P-generic, $\mathcal{T}, b \in P[g]$, and $P[g]$ satisfies "$\mathcal{T} ^\frown b$ is via Γ, verified by background constructions". The iterability of $P[g]$ above $\text{rank}(g)$ ensures the iterability of the Q-structures that $P[g]$ builds to verify $\mathcal{T} ^\frown b$ (above the relevant δ's). Therefore $\mathcal{T} ^\frown b$ is via Γ.)

So in $N[G]$, $\Sigma_S = \Gamma \restriction \text{HC}^{N[G]}$, as required. For the final hypothesis of 6.2, if α is as there, $\rho_1^{N|\alpha+\omega} = \omega$ by 2.4. So let $M \triangleleft N|\omega_1^N$ with $\rho_1^{\mathcal{J}_1(M)} = \omega$. Then Γ follows the unique $(0, \omega_1{+}1)$-strategy for $\mathcal{J}_1(M)$. So by 2.4, Γ induces the unique $(\omega, \omega_1 + 1)$-strategy for M. So if $\gamma \in \text{OR}^N$ and $\pi : M \to N|\gamma$ is elementary, Γ induces the strategy induced by π. The final hypothesis of 6.2 follows. □

6.4. Definition. Let $Z \in \mathcal{P}(\mathbb{R})^{M_\omega}$ and $z \in \mathbb{R}^{M_\omega}$. Then Z is *correctly* $(\Delta^2_1(z))^{L(\mathbb{R})}$ iff there is a $(\Delta^2_1(z))^{L(\mathbb{R})}$ set Z' such that $Z = Z' \cap M_\omega$.

If $z \in \mathbb{R}^{M_\omega}$, then by [20, 7.15], M_ω can compute $L(\mathbb{R})$ truth about z, and therefore "correctly $(\Delta^2_1(z))^{L(\mathbb{R})}$" is definable over M_ω, uniformly from only z. Moreover, by [20, 7.20], if φ is $\Sigma_1^{L(\mathbb{R})}$ and $L(\mathbb{R}) \models \varphi(z)$, then $L(\mathbb{R})^{M_\omega} \models \varphi(z)$. So 6.4 is a reasonable analogue of 1.1.

The following was observed by Steel and Woodin in the case of $N = M_\omega$; the author then generalized this result to the other cases.

6.5. Theorem (Steel, Woodin, S.). *Let N, λ be as in 6.3. In N, the classes (a) homogeneously Suslin, and (b) $<\lambda$-homogeneously Suslin, coincide. If $N = M_\omega$, these further coincide, in M_ω, with the classes (c) $\delta_0 + 1$-universally Baire, and (d) correctly $(\underset{\sim}{\Delta}{}^2_1)^{L(\mathbb{R})}$.*

Proof. By [6, 3.3.13], (b) coincides with (e) λ-universally Baire.

The preceding two lemmas show "(a) ⇒ (e)", so we have "(a) ⇔ (b)".

For "(c) ⇒ (d)" and "(d) ⇒ (e)", combine the proofs of [20, 7.20] and 4.1, using [20, 7.13]. □

6.6. Lemma. *Let N, λ, S satisfy the hypotheses of 6.2, other than the requirement that N be tame. Assume N has a Woodin cardinal. Then in N, every δ_0^N-hom set is λ-universally Baire.*

Proof. We may assume $\delta_0^N < \lambda$. Let θ, η, M, π be as usual. We may assume $S, \lambda \in \text{rg}(\pi)$. Let $\eta^* = (\eta^+)^N$ and $M^* = \text{Hull}_\omega^{N|\eta^*}(\emptyset)$ and $\pi^* : M^* \to N|\eta^*$ be elementary. Let $\nu \in \text{rg}(\pi) \cap N|\theta$ be a δ_0-complete homogeneity system. In N, we will define λ-absolutely complementing trees W, I such that $p[W] = p[\nu]$.

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[24]For the case of $G = \emptyset$, an alternative, detailed proof, which gives more information, is given in [14, 5.14]. The following generalizes that proof to $G \neq \emptyset$. We work in V. Fix a sequence $X = \langle x_\alpha \rangle_{\alpha<\omega_1}$, with x_α a real coding α, and require of all x-certificates $\pi : M \to V_\theta$ that M be countable and $X \in \text{rg}(\pi)$. This ensures that if $N = \mathfrak{C}_\omega(N_\alpha^{\mathbb{C}})$ and $\rho + 1$ is a cutpoint of $N_\alpha^{\mathbb{C}}$, then there's a fixed $N_\alpha^{\mathbb{C}}$-generic G for $\text{Col}(\omega, \rho)$ that's in $\text{dom}(\pi)$ for eventually all N-certificates π. Then the proof of [14, 5.14] goes through with all trees $\mathcal{T} \in N_\alpha[G]$.

Let W be the tree on $\omega \times \eta^*$ building $(x, P, \tau, g, \mathcal{T}, Q)$ such that $\tau : P \to N|\eta^*$ is elementary, g is $<\lambda^P$-generic over P, $M^* \in P$, $x \in \mathbb{R}^{P[g]}$, and $P[g]$ satisfies "$\mathcal{T}$ is a normal tree on M^* via Σ_{S^P}, with last model Q, $i^{\mathcal{T}}$ exists, x is Q-generic for Q's extender algebra at δ_0^Q, and $\mathrm{Ult}(Q, \nu_x^Q)$ is wellfounded" [25] . Let I be defined in the same way, but replacing "wellfounded" with "illfounded".

If G is $< \lambda$-generic over N, then clearly $\mathbb{R}^{N[G]} \subseteq p[W] \cup p[I]$, since $N[G]$ successfully computes genericity iterations on M^*.

Let us verify that in N, $p[W] \subseteq p[\nu]$ and $p[I] \subseteq \mathbb{R} - p[\nu]$. It will also follow that in V, $p[W] \cap [I] = \emptyset$, so this will complete the proof. [26]

Work in N. Let $x \in p[I]$ (or $x \in p[W]$), witnessed by $(P, \tau, g, \mathcal{T}, Q)$. So $\mathrm{Ult}(Q, \nu_x^Q)$ is illfounded (or wellfounded). As in 6.2, $\mathcal{T}$ is via Σ_S, so $\pi^*\mathcal{T}$ has wellfounded models. Since $M^*, \mathcal{T}$ are countable in N, there's an elementary $\sigma : Q \to N|\eta^*$, and $\sigma \circ i_{\mathcal{T}} = \pi^*$.

Assume $x \in p[I]$, so $\mathrm{Ult}(Q, \nu_x^Q)$ is illfounded. Note $\sigma``\nu_x^Q = \nu_x$, so $\mathrm{Ult}(Q, \nu_x^Q)$ embeds into $\mathrm{Ult}(N|\eta^*, \nu_x)$, so the latter is illfounded. So $x \notin p[\nu]$, as required.

Now assume $x \in p[W]$. Note $M \in Q$ and $M = \mathrm{Hull}_\omega^{Q|\eta^Q}(\emptyset)$. Let $\bar{\nu} = \nu^Q$. By δ_0^Q-completeness, $\bar{\nu}$ extends to a homogeneity system $\bar{\nu}^+$ of $Q[x]$, and $\mathrm{Ult}(Q[x], \bar{\nu}_x^+) = \mathrm{Ult}(Q, \bar{\nu}_x)[x]$. Since $x \in p[W]$, these are wellfounded. Now $\mathrm{Ult}(Q[x], \bar{\nu}_x^+)$ sees an embedding from $\bar{U} = \mathrm{Ult}(M, \nu_x^M)$ into $i_{\bar{\nu}_x^+}^{Q[x]}(Q|\eta^Q)$. Therefore in $Q[x]$, $\bar{U}$ embeds into $Q|\eta^Q$. So in N, $\bar{U}$ embeds into $\sigma(Q|\eta^Q)$. As in Claim 2 of 6.2, this implies that in N, $\bar{U}$ embeds into some $N' \lhd N|\omega_1^N$, so $\bar{U}$ is λ-iterable in N. So as in Claim 3 of 6.2, $\mathrm{Ult}(N, \nu_x)$ is wellfounded. So $x \in p[\nu]$. □

We can apply this, for example, to M_{wlim} (see [18, 3.5]).

6.7. Theorem. *In M_{wlim}, every δ_0-hom set is $<\lambda$-hom, where λ is the sup of the Woodins.*

Proof. The proof of "(b) ⇒ (c)" in [18, 5.1] shows 6.6 applies. □

6.8. Question. In M_{wlim}, are all hom sets $<\lambda$-hom?

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[25] See footnote 21.

[26] We could have defined W, I so as to require that $\mathcal{T}$ be *the* genericity iteration, via Σ_S, making x generic, which would immediately ensure that $p[W] \cap p[I] = \emptyset$, but this wouldn't actually change the remainder of the proof.

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