AN ALMOST KUREPA SUSLIN TREE WITH STRONGLY NON-SATURATED SQUARE

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ABSTRACT. For uncountable downwards closed subtrees U and W of an ω_1-tree T, we say that U and W are *strongly almost disjoint* if their intersection is a finite union of countable chains. The tree T is *strongly non-saturated* if there exists a strongly almost disjoint family of ω_2-many uncountable downwards closed subtrees of T. In this article we construct a c.c.c. forcing which adds a Suslin tree together with a family of ω_2-many strongly almost disjoint automorphisms of it (and thus the square of the Suslin tree is strongly non-saturated). To achieve this goal, we introduce a new idea called *ρ-separation*, which is an adaptation to the finite context of the notion of separation which was recently introduced by Stejskalová and the first author for the purpose of adding automorphisms of a tree with a forcing with countable conditions.

CONTENTS

1. INTRODUCTION

The consistency of the existence of a Suslin tree was originally proven independently by Jech [Jec67] and Tennenbaum [Ten68] using the technique of forcing. In Jech's forcing, conditions are countable initial segments of the generic tree with a top level, whereas Tennenbaum's forcing consists of finite approximations of the generic tree. Jech's forcing is countably closed and Tennenbaum's forcing is c.c.c. As a variation of his forcing for adding

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a Suslin tree, Jech [Jec72] defined a countably closed forcing which adds a Suslin tree together with κ-many automorphisms of it, where κ is any infinite cardinal number satisfying that $\kappa^\omega = \kappa$. While it is not mentioned explicitly in his article, the automorphisms added by Jech's forcing are almost disjoint in the sense that any two of them agree on only countably many elements of the tree. As a consequence of this fact, if $\kappa \geq \omega_2$ then the generic Suslin tree is an *almost Kurepa Suslin tree*, which means that forcing with the Suslin tree turns it into a Kurepa tree. Namely, applying the automorphisms to a generic branch produces κ-many distinct cofinal branches.

Another application of Jech's forcing for adding a Suslin tree is due to Stewart [Ste66], who designed a countably closed forcing which adds a Kurepa tree. This fact suggests the question of whether there is a variation of Tennenbaum's c.c.c. forcing for adding a Suslin tree which adds a Kurepa tree. Jensen and Schlechta [JS90] proved that this is not always possible: after forcing with the Lévy collapse to turn a Mahlo cardinal into ω_2, there does not exist a c.c.c. forcing which adds a Kurepa tree. On the other hand, Jensen proved that if $\square_{\omega_1}$ holds then there exists a c.c.c. forcing which adds a Kurepa tree. Later, Veličković [Vel92] proved the same result with a simpler argument using the function ρ of Todorčević [Tod87], whose existence follows from $\square_{\omega_1}$. Other examples of using ρ to define c.c.c. forcings were given later by Todorčević [Tod07, Chapter 7].

In light of these forcing constructions, a natural question is whether it is consistent that there exists a c.c.c. forcing which adds an almost Kurepa Suslin tree. Note that the Jensen-Schlechta limitation mentioned above also applies to this problem since an almost Kurepa Suslin tree is a c.c.c. forcing which adds a Kurepa tree. The main result of this article is that if $\square_{\omega_1}$ holds then there exists a c.c.c. forcing with finite conditions which adds a Suslin tree with ω_2-many almost disjoint automorphisms. In fact, the family of automorphisms satisfies a very strong form of almost disjointness which we introduce next.

Recall that if U and W are uncountable downwards closed subtrees of an ω_1-tree T, then U and W are *almost disjoint* if $U \cap W$ is countable. Observe that if b and c are distinct cofinal branches of T, then b and c are uncountable downwards closed subtrees of T which are almost disjoint. In fact, $b \cap c$ is a countable chain, and if T is normal then $b \cap c$ is equal to the chain of elements less than or equal to the meet of b and c. This example of cofinal branches suggests stronger forms of almost disjointness for uncountable downwards closed subtrees U and W of T. We could ask for $U \cap W$ to be a union of finitely many countable chains, or the slightly stronger property that $U \cap W$ is contained in the downward closure of a finite subset of T. Let us say that U and W are *strongly almost disjoint* if $U \cap W$ is a finite union of countable chains. Note that an automorphism of T is an uncountable downwards closed subtree of the tree product $T \otimes T$, so we can talk about strongly almost disjoint automorphisms of T considered as subtrees of $T \otimes T$.

König, Larson, Moore, and Veličković [KLMV08] introduced the idea of a *saturated Aronszajn tree*, which is an Aronszajn tree satisfying that any family of almost disjoint uncountable downwards closed subtrees of it has size at most ω_1. A standard example of a non-saturated Aronszajn tree, due to Todorčević (see [Bau85, Section 2]), is the tree product $T \otimes K$, where T is any Aronszajn tree and K is a Kurepa tree. Namely, if $\{b_\alpha : \alpha < \omega_2\}$ is a family of cofinal branches of K, then letting $U_\alpha = T \otimes b_\alpha$ for each $\alpha < \omega_2$, the family $\{U_\alpha : \alpha < \omega_2\}$ is a witness that $T \otimes K$ is non-saturated.

With the above strengthening of almost disjointness at hand, we introduce the idea of an ω_1-tree T being *strongly non-saturated*, by which we mean that there exists a family of size at least ω_2 consisting of strongly almost disjoint uncountable downwards closed subtrees of T. Since a Kurepa tree is strongly non-saturated, as witnessed by the family of its cofinal branches, a strongly non-saturated ω_1-tree is a generalization of a Kurepa tree. It is not hard to show that if there exists a strongly non-saturated Aronszajn tree, then CH fails. The non-saturated Aronszajn tree from the previous paragraph is not strongly non-saturated. Namely, since T is Aronszajn, we can fix a countable ordinal γ such that T_γ is infinite. Find α and β such that the first ordinal δ for which b_α and b_β are different on level δ is greater than γ. Let x be the element of b_α and b_β with height γ. Then $U_\alpha \cap U_\beta$ contains the infinite antichain $T_\gamma \times \{x\}$, and therefore is not a finite union of countable chains. Since none of the known examples of non-saturated Aronszajn trees are strongly non-saturated, a natural question is whether it is consistent that there exists a strongly non-saturated Aronszajn tree.

Theorem. *Assuming $\square_{\omega_1}$, there exists a c.c.c. forcing $\mathbb{P}$ which adds a Suslin tree together with a family of ω_2-many strongly almost disjoint automorphisms of it. So $\mathbb{P}$ forces the existence of an almost Kurepa Suslin tree T such that $T \otimes T$ is a strongly non-saturated Aronszajn tree.*

The main technique used in this article is *ρ-separation*, which is a variation of the notion of separation recently introduced by Stejskalová and the first author [KS] for the purpose of adding automorphisms to an ω_1-tree by forcing with countable conditions. By working the function ρ into the definition of separation, we are able to adapt many of the key tools of [KS] to the finite context.

We assume that the reader has a background in ω_1-trees and forcing. Our notation is standard; we refer the reader to [KS, Section 1] for basic terminology and definitions concerning trees.

2. Standard Finite Trees

The goal of this article is to define a forcing which adds a Suslin tree with some remarkable properties by a c.c.c. forcing with finite conditions. While the tree does not exist in the ground model, it is helpful to specify the levels of the tree there. Specifically, every member of level $\alpha < \omega_1$ of the generic tree is some ordinal γ such that $\omega \cdot \alpha \leq \gamma < \omega \cdot (\alpha + 1)$. With this in mind, we define the *height* of a countable ordinal γ to be the unique ordinal α such that $\omega \cdot \alpha \leq \gamma < \omega \cdot (\alpha + 1)$, and we denote the height of γ by $\mathrm{ht}(\gamma)$.

Our forcing poset consists of conditions with two components, where the first component is a finite tree and the second component is a finite indexed family of functions defined on the tree. In this section we develop some basic ideas about the finite trees which appear in our conditions.

Definition 2.1. *A* standard finite tree *is a pair $(T, <_T)$ satisfying:*

(1) *T is a finite subset of $\{0\} \cup (\omega_1 \setminus \omega)$ and $0 \in T$;*
(2) *$<_T$ is a tree ordering on T, meaning a strict partial-ordering such that for any $x \in T$, the set $\{y \in T : y <_T x\}$ is linearly ordered by $<_T$;*
(3) *if $x <_T y$ then $\mathrm{ht}(x) < \mathrm{ht}(y)$;*

(4) *for all* $x \in T$ *and for all* $\alpha \in \{\text{ht}(z) : z \in T\} \cap \text{ht}(x)$, *there exists some* $y \in T$ *such that* $\text{ht}(y) = \alpha$ *and* $y <_T x$.

Suppose that $(T, <_T)$ is a standard finite tree. We oftentimes abbreviate $(T, <_T)$ by just T. Define $\text{ht}[T] = \{\text{ht}(x) : x \in T\}$. For all $\alpha \in \text{ht}[T]$, define $T_\alpha = \{x \in T : \text{ht}(x) = \alpha\}$. Note that T_α is an antichain of T by Definition 2.1(3). If $x \in T_\alpha$ and $\beta \in \text{ht}[T] \cap \alpha$, we write $x \restriction_T \beta$ (or just $x \restriction \beta$ is T is understood from context) for the unique $y \in T_\beta$ such that $y <_T x$, and if $X \subseteq T_\alpha$, define $X \restriction \beta = \{x \restriction \beta : x \in X\}$. For such a set X, we say that X has *unique drop-downs to* β if the map $x \mapsto x \restriction \beta$ on X is injective. If $x <_T y$, then we say that y is a *successor* of x in T, and if $x <_T y$ and $\text{ht}(y) = \min(\text{ht}[T] \setminus (\text{ht}(x)+1))$, then we say that y is an *immediate successor* of x in T. The set of successors of x in T is denoted by $\text{Succ}_T(x)$ and the set of immediate successors of x in T is denoted by $\text{ISucc}_T(x)$.

If T and U are standard finite trees, we say that U is an *extension* of T (or U *extends* T) if $T \subseteq U$ and $<_T \subseteq <_U$. We claim that if U extends T, then U *end-extends* T in the sense that $<_T = <_U \cap (T \times T)$. For if not, then there are distinct $x, y \in T$ such that $x \not<_T y$ but $x <_U y$. By Definition 2.1(4), we can fix $z \in T$ such that $\text{ht}_T(z) = \text{ht}_T(x)$ and $z <_T y$. Then in U, z and x are both below y but are incomparable since they have the same height, which contradicts that $<_U$ is a tree ordering.

Let T be a standard finite tree. Observe that by Definition 2.1(1), 0 is the unique element of T with height 0, and by Definition 2.1(4), $0 \leq_T x$ for all $x \in T$. In other words, every standard finite tree has 0 as a root. For any set $Y \subseteq T$, define the *downward closure* of Y to be the set $\{z \in T : \exists x \in Y \ z \leq_T x\}$. For any $x, y \in T$, let $x \wedge_T y$ (or just $x \wedge y$ if T is understood from context) denote the $<_T$-largest element z of T such that $z \leq_T x$ and $z \leq_T y$. Note that $x \wedge y$ exists since T has a root. A simple fact which is useful below is that if x and y are in T, $\alpha \leq \text{ht}(x), \text{ht}(y)$, and $x \restriction \alpha \neq y \restriction \alpha$, then $x \wedge y = (x \restriction \alpha) \wedge (y \restriction \alpha)$.

Definition 2.2. *If* T *and* U *are standard finite trees, we say that* U *is a* simple extension *of* T *if:*

- U *is an extension of* T*;*
- $U \setminus T \subseteq \bigcup\{U_\alpha : \alpha \in \text{ht}[U] \setminus \text{ht}[T]\}$*;*
- *if* $\alpha \in \text{ht}[U] \setminus \text{ht}[T]$ *is less than* $\max(\text{ht}[T])$ *and* β *is the least element of* $\text{ht}[T]$ *greater than* α*, then* T_β *has unique drop-downs to* α.

Note that in the third bullet point, $T_\beta = U_\beta$ so U_β has unique drop-downs to α.

We leave the easy proofs of the next two lemmas to the reader.

Lemma 2.3. *The relation on the set of all standard finite trees of being a simple extension is transitive.*

Lemma 2.4. *Suppose that* T *and* U *are standard finite trees and* U *is a simple extension of* T*. Then for all* a *and* b *in* T*,* $a \wedge_T b = a \wedge_U b$.

Lemma 2.5. *Suppose that* T *and* U *are standard finite trees,* U *is a simple extension of* T*,* $\alpha \in \text{ht}[U] \setminus \text{ht}[T]$ *is less than* $\max(\text{ht}[T])$*, and* $\beta = \min(\text{ht}[T] \setminus (\alpha + 1))$*. Assume that* $a_0 \in U_\alpha$*,* a_0^+ *is the unique element of* T_β *above* a_0*,* $a_1 \in T$*, and* a_0^+ *and* a_1 *are incomparable in* T*. Then* $a_0 \wedge_U a_1 = a_0^+ \wedge_T a_1$.

Proof. By Lemma 2.4, $a_0^+ \wedge_T a_1 = a_0^+ \wedge_U a_1$. First, assume that $\mathrm{ht}_T(a_1) < \alpha$. Since a_0^+ and a_1 are incomparable in T, a_1 is not below a_0 in U. So $a_1 \neq a_0^+ \upharpoonright_T \mathrm{ht}(a_1) = a_0 \upharpoonright_U \mathrm{ht}(a_1)$, and $a_0 \wedge_U a_1$ and $a_0^+ \wedge_T a_1$ are both equal to $(a_0^+ \upharpoonright_T \mathrm{ht}(a_1)) \wedge_U a_1$. Secondly, assume that $\mathrm{ht}_T(a_1) > \alpha$. By the minimality of β, $\mathrm{ht}_T(a_1) \geq \beta$. Since a_1 is incomparable with a_0^+ in T, $a_1 \upharpoonright_T \beta \neq a_0^+$. As U is a simple extension of T, $a_1 \upharpoonright_U \alpha = (a_1 \upharpoonright_T \beta) \upharpoonright_U \alpha \neq a_0$. So $a_0 \wedge_U a_1$ and $a_0^+ \wedge_T a_1$ are both equal to $a_0 \wedge_U (a_1 \upharpoonright_U \alpha)$. □

Lemma 2.6. *Suppose that T is a standard finite tree and $B \subseteq \omega_1$ is a finite set such that* $\mathrm{ht}[T] \subseteq B$. *Then there exists a standard finite tree U which is a simple extension of T such that* $\mathrm{ht}[U] = B$.

Proof. We prove that if T is a standard finite tree and $\alpha \in \omega_1 \setminus \mathrm{ht}[T]$, then there exists a standard finite tree T^+ which is a simple extension of T such that $\mathrm{ht}[T^+] = \mathrm{ht}[T] \cup \{\alpha\}$. Once we prove this statement, the lemma follows easily by induction on the size of $B \setminus A$ using Lemma 2.3. If $\alpha > \max(\mathrm{ht}[T])$, pick some $x \in T$ with height equal to $\max(\mathrm{ht}[T])$, and define T^+ by adding an immediate successor of x with height α. Suppose that $\alpha < \max(\mathrm{ht}[T])$. Since $0 \in \mathrm{ht}[T]$, we can fix successive elements $\beta < \delta$ of $\mathrm{ht}[T]$ such that $\beta < \alpha < \delta$. Fix some injective mapping $x \mapsto x^-$ from T_δ into $\{z \in \omega_1 : \mathrm{ht}(z) = \alpha\}$. Define the underlying set of T^+ to be equal to $T \cup \{x^- : x \in T_\delta\}$. Let $<_{T^+}$ consist of the relations in $<_T$, together with the new relations $x^- <_{T^+} z$ whenever $x \leq_T z$ and $y <_{T^+} x^-$ whenever $y \leq_T x \upharpoonright \beta$. □

Definition 2.7. *A standard finite tree T is* normal *if for all $x \in T$ and for all $\alpha \in \mathrm{ht}[T] \setminus (\mathrm{ht}(x) + 1)$, there exists some $y \in T$ such that $\mathrm{ht}(y) = \alpha$ and $x <_T y$.*

Definition 2.8. *A standard finite tree T is* Hausdorff *if for every limit ordinal $\delta \in \mathrm{ht}[T]$, letting $\beta = \max(\mathrm{ht}[T] \cap \delta)$, T_δ has unique drop-downs to β.*

Given a standard finite tree T and an element x below the top level of T, it is a simple matter to define an extension U of T such that $\mathrm{ht}[U] = \mathrm{ht}[T]$, $U \setminus T \subseteq \mathrm{ISucc}_U(x)$, and $\mathrm{ISucc}_U(x)$ is as large as you want. In particular, repeating this process and working our way up the levels of the tree, we can build a normal extension of T. The next two lemmas follow from this observation.

Lemma 2.9. *Suppose that T is a standard finite tree. Then there exists a standard finite tree U which extends T such that* $\mathrm{ht}[U] = \mathrm{ht}[T]$ *and U is normal.*

Lemma 2.10. *Suppose that T is a standard finite tree, $\alpha \in \mathrm{ht}[T] \cap \max(\mathrm{ht}[T])$, $X \subseteq T_\alpha$, and n is a positive natural number such that each element of X has at most n-many immediate successors in T. Then there exists a standard finite tree U which extends T such that:*

- $\mathrm{ht}[T] = \mathrm{ht}[U]$;
- $U \setminus T \subseteq \bigcup\{\mathrm{ISucc}_U(x) : x \in X\}$;
- *every element of X has exactly n-many immediate successors.*

3. STANDARD FUNCTIONS

The second component of a condition in our forcing is a finite indexed family of partial functions defined on its standard finite tree. In this section we introduce and analyze some basic ideas concerning such functions.

Definition 3.1. *Let T be a standard finite tree and let f be a partial function from T to T.*

- *f is* strictly increasing *if for all $x, y \in \text{dom}(f)$, if $x <_T y$ then $f(x) <_T f(y)$;*
- *f is* level preserving *if for all $x \in \text{dom}(f)$, $\text{ht}(x) = \text{ht}(f(x))$;*
- *if f is level preserving and strictly increasing, then f is* downwards closed in T *if whenever $x \in \text{dom}(f)$ and $\beta \in \text{ht}[T] \cap \text{ht}(x)$, then $x \upharpoonright \beta \in \text{dom}(f)$.*

Definition 3.2. *Let T be a standard finite tree. A partial function f from T to T is called a* standard function *on T if it is injective, strictly increasing, level preserving, downwards closed, and satisfies that for all $x \in \text{dom}(f) \setminus \{0\}$, $f(x) \neq x$.*

Lemma 3.3. *Let T be a standard tree and let f be a standard function on T. Then f^{-1} is strictly increasing. Moreover, if c and d are in the domain of f and are incomparable in T, then $f(c)$ and $f(d)$ are also incomparable in T.*

Proof. Let $c, d \in \text{dom}(f)$ and assume that $f(c) <_T f(d)$. We claim that $c <_T d$. Since f is level preserving, $\text{ht}(c) = \text{ht}(f(c))$ and $\text{ht}(d) = \text{ht}(f(d))$. So $\text{ht}(c) < \text{ht}(d)$. Suppose for a contradiction that $c \not<_T d$. Then $d \upharpoonright \text{ht}(c) \neq c$. Since f is strictly increasing, $f(d \upharpoonright \text{ht}(c)) <_T f(d)$. So $f(c)$ and $f(d \upharpoonright \text{ht}(c))$ are both equal to $f(d) \upharpoonright \text{ht}(c)$, which contradicts that f is injective.

Now assume that $c, d \in \text{dom}(f)$ are incomparable in T. If c and d have the same height, then they are different. Since f is injective and level preserving, $f(c)$ and $f(d)$ are distinct and have the same height, and hence are incomparable in T. Assume without loss of generality that $\text{ht}(c) < \text{ht}(d)$. Since f is level preserving, $\text{ht}(f(c)) < \text{ht}(f(d))$. So if $f(c)$ and $f(d)$ are comparable in T, then $f(c) <_T f(d)$. Since f^{-1} is strictly increasing, $c <_T d$, which is a contradiction. □

Lemma 3.4. *Suppose that T and U are standard finite trees, U is an extension of T, $\text{ht}[U] \cap (\max(h[T]) + 1) = \text{ht}[T]$, and F is a standard function on T. Then F is a standard function on U.*

Proof. Clearly, F is injective, strictly increasing, level preserving, and has no fixed points other than 0 (considered as a partial function from U to U). The assumption about the height function easily implies that F is downwards closed in U. □

For a standard finite tree T, define $T \otimes T$ to be the set of all pairs (a, b) such that for some $\alpha \in \text{ht}[T]$, a and b are in T_α. A set $Z \subseteq T \otimes T$ is *downwards closed* if for all $(a, b) \in Z$, if $(c, d) \in T \otimes T$, $c \leq_T a$, and $d \leq_T b$, then $(c, d) \in Z$. Note that a partial function f from T to T is level preserving iff $f \subseteq T \otimes T$. For any partial level preserving function f from T to T, the *downward closure* of f is the set of all $(a, b) \in T \otimes T$ such that for some $x \in \text{dom}(f)$, $a \leq_T x$ and $b \leq_T f(x)$.

Lemma 3.5. *Let T be a standard finite tree and let $f \subseteq T \otimes T$ be downwards closed. Then f is a strictly increasing function iff for all pairs (a_0, b_0) and (a_1, b_1) in f, $\text{ht}(a_0 \wedge a_1) \leq \text{ht}(b_0 \wedge b_1)$.*

Proof. For the reverse implication, we prove the contrapositive: if f is not a strictly increasing function, then there exist (a_0, b_0) and (a_1, b_1) in f such that $\mathrm{ht}(b_0 \wedge b_1) < \mathrm{ht}(a_0 \wedge a_1)$. If f is not a strictly increasing function, then it is either not a function or it is a function but it is not strictly increasing. Suppose that f is not a function. Then there exist a, b_0, and b_1 in T such that $b_0 \neq b_1$ and (a, b_0) and (a, b_1) are both in f. Since b_0 and b_1 have the same height as a, and hence the same height as each other, they are incomparable in T. So $\mathrm{ht}(b_0 \wedge b_1) < \mathrm{ht}(b_0) = \mathrm{ht}(a) = \mathrm{ht}(a \wedge a)$, and the pairs (a, b_0) and (a, b_1) are as desired. Now suppose that f is a function but it is not strictly increasing. Then there are $c <_T d$ such that $f(c) \not<_T f(d)$. Note that $\mathrm{ht}(f(c)) = \mathrm{ht}(c) < \mathrm{ht}(d) = \mathrm{ht}(f(d))$, and therefore $f(c)$ and $f(d)$ are incomparable in T. So $c \wedge d = c$ and $f(c) \wedge f(d) <_T f(c)$. Hence, the pairs $(c, f(c))$ and $(d, f(d))$ are as required.

For the forward implication, we prove that if f is a strictly increasing function, then for any (a_0, b_0) and (a_1, b_1) in f, $\mathrm{ht}(a_0 \wedge a_1) \leq \mathrm{ht}(b_0 \wedge b_1)$. In other words, we prove that for all a_0 and a_1 in the domain of f, $\mathrm{ht}(a_0 \wedge a_1) \leq \mathrm{ht}(f(a_0) \wedge f(a_1))$. Since f is downwards closed, $a_0 \wedge a_1 \in \mathrm{dom}(f)$, and as f is strictly increasing, $f(a_0 \wedge a_1) \leq_T f(a_0), f(a_1)$. By the definition of meet, $f(a_0 \wedge a_1) \leq_T f(a_0) \wedge f(a_1)$. Hence, $\mathrm{ht}(a_0 \wedge a_1) = \mathrm{ht}(f(a_0 \wedge a_1)) \leq \mathrm{ht}(f(a_0) \wedge f(a_1))$. □

Lemma 3.6. *Let T be a standard finite tree and let f be a strictly increasing, level preserving, and downwards closed partial function from T to T. Then f is injective iff for all a_0 and a_1 in the domain of f,* $\mathrm{ht}(a_0 \wedge a_1) = \mathrm{ht}(f(a_0) \wedge f(a_1))$.

Proof. By Lemma 3.5, for all a_0 and a_1 in the domain of f, $\mathrm{ht}(a_0 \wedge a_1) \leq \mathrm{ht}(f(a_0) \wedge f(a_1))$. For the reverse implication, we prove the contrapositive: suppose that f is not injective, and we find a_0 and a_1 in the domain of f such that $\mathrm{ht}(a_0 \wedge a_1) < \mathrm{ht}(f(a_0) \wedge f(a_1))$. Since f is not injective, there are distinct a_0 and a_1 and some b such that $f(a_0) = b$ and $f(a_1) = b$. Since a_0 and a_1 have the same height as b, and hence the same height as each other, they are incomparable in T. So $\mathrm{ht}(a_0 \wedge a_1) < \mathrm{ht}(a_0) = \mathrm{ht}(b) = \mathrm{ht}(f(a_0) \wedge f(a_1))$.

For the forward implication, suppose that f is injective and we show that for all a_0 and a_1 in the domain of f, $\mathrm{ht}(a_0 \wedge a_1) \geq \mathrm{ht}(f(a_0) \wedge f(a_1))$. Suppose for a contradiction that $\mathrm{ht}(a_0 \wedge a_1) < \mathrm{ht}(f(a_0) \wedge f(a_1))$. Let $\xi = \mathrm{ht}(f(a_0) \wedge f(a_1))$. Then $\mathrm{ht}(a_0 \wedge a_1) < \xi$. Note that $\xi \leq \mathrm{ht}(f(a_0)) = \mathrm{ht}(a_0)$ and $\xi \leq \mathrm{ht}(f(a_1)) = \mathrm{ht}(a_1)$, so $a_0 \restriction \xi$ and $a_1 \restriction \xi$ are defined. As $\mathrm{ht}(a_0 \wedge a_1) < \xi$, $a_0 \restriction \xi \neq a_1 \restriction \xi$. Since f is downwards closed, $a_0 \restriction \xi$ and $a_1 \restriction \xi$ are in the domain of f, and as f is strictly increasing and level preserving, $f(a_0 \restriction \xi) = f(a_0) \restriction \xi = f(a_0) \wedge f(a_1) = f(a_1) \restriction \xi = f(a_1 \restriction \xi)$, which contradicts that f is injective. □

The next lemma follows immediately from the previous one.

Lemma 3.7. *Let T be a standard finite tree and let f be a standard function on T. Then for all a_0 and a_1 in the domain of f,* $\mathrm{ht}(a_0 \wedge a_1) = \mathrm{ht}(f(a_0) \wedge f(a_1))$.

Lemma 3.8. *Let T be a standard finite tree and let f be a standard function on T. Suppose that U is a simple extension of T. Let g be the downward closure of f in U. Then:*

- *g is a standard function on U;*
- *$g \restriction T = f$;*

- *if* $\alpha \in \mathrm{ht}[U] \setminus \mathrm{ht}[T]$ *is less than* $\max(\mathrm{ht}[T])$ *and* β *is the least element of* $\mathrm{ht}[T]$ *greater than* α*, then for all* $x, y \in U_\beta$*,* $g(x) = y$ *iff* $g(x \restriction \alpha) = y \restriction \alpha$*.*

Proof. We leave it to the reader to verify the second and third bullet points, which is straightforward. We prove that g is a standard function on U. It suffices to prove this in the special case that $\mathrm{ht}[U] \setminus \mathrm{ht}[T]$ is a singleton. For then we can prove the general statement by induction on the size of $\mathrm{ht}[U] \setminus \mathrm{ht}[T]$. So let $\mathrm{ht}[U] \setminus \mathrm{ht}[T]$ consist of one ordinal α. If $\alpha > \max(\mathrm{ht}[T])$, then $f = g$, and easily f is a standard function on U. Assume that $\alpha < \max(\mathrm{ht}[T])$ and let $\beta = \min(\mathrm{ht}[T] \setminus (\alpha + 1))$.

Since f is downwards closed, if $(a, b) \in g \setminus f$, then a and b have height α and hence neither of them are in T. So $(a, b) \in g \setminus f$ iff a and b have height α and there are a^+ and b^+ in T_β above a and b respectively such that $f(a^+) = b^+$. If $a = b$, then since U is a simple extension of T, $a^+ = b^+$, contradicting the fact that f is a standard function. So assuming that g is a function, it has no fixed points other than 0.

By definition, $g \subseteq U \otimes U$ is downwards closed, so if g is a function then it is obviously level preserving. So it suffices to prove that g is an injective strictly increasing function. By Lemmas 3.5 and 3.6, it suffices to prove that whenever (a_0, b_0) and (a_1, b_1) are in g, then $\mathrm{ht}(a_0 \wedge_U a_1) = \mathrm{ht}(b_0 \wedge_U b_1)$. We consider three cases.

Case 1: (a_0, b_0) and (a_1, b_1) are both in f. By Lemmas 2.4 and 3.7, $\mathrm{ht}(a_0 \wedge_U a_1) = \mathrm{ht}(a_0 \wedge_T a_1) = \mathrm{ht}(b_0 \wedge_T b_1) = \mathrm{ht}(b_0 \wedge_U b_1)$.

Case 2: (a_0, b_0) and (a_1, b_1) are both in $g \setminus f$. Then a_0, b_0, a_1, and b_1 are in U_α. Fix a_0^+, b_0^+, a_1^+, and b_1^+ in T_β above a_0, b_0, a_1, and b_1 respectively such that $f(a_0^+) = b_0^+$ and $f(a_1^+) = b_1^+$. First, assume that $a_0 \neq a_1$. Then $a_0^+ \neq a_1^+$, and since f is injective, $b_0^+ \neq b_1^+$. By unique drop-downs, $b_0 \neq b_1$. Hence, $a_0 \wedge_U a_1 = a_0^+ \wedge_T a_1^+$ and $b_0 \wedge_U b_1 = b_0^+ \wedge_T b_1^+$. So $\mathrm{ht}(a_0 \wedge_U a_1) = \mathrm{ht}(a_0^+ \wedge_T a_1^+)$, which by Lemma 3.7 is equal to $\mathrm{ht}(b_0^+ \wedge_T b_1^+) = \mathrm{ht}(b_0 \wedge_U b_1)$. Secondly, assume that $a_0 = a_1$. By unique drop-downs, $a_0^+ = a_1^+$. So $b_0^+ = b_1^+$, and hence $b_0 = b_1$. Therefore, $\mathrm{ht}(a_0 \wedge a_1) = \alpha = \mathrm{ht}(b_0 \wedge b_1)$.

Case 3: One of (a_0, b_0) and (a_1, b_1) is in $g \setminus f$ and the other is in f. Without loss of generality, assume that (a_0, b_0) is in $g \setminus f$ and (a_1, b_1) is in f. Then a_0 and b_0 are in U_α, and there are a_0^+ and b_0^+ above a_0 and b_0 respectively with height β such that $f(a_0^+) = b_0^+$. If $a_1 \geq_T a_0^+$, then since f is strictly increasing, $b_1 \geq_T b_0^+$. Hence, $a_0 <_U a_1$ and $b_0 <_U b_1$. So $\mathrm{ht}(a_0 \wedge_U a_1) = \mathrm{ht}(a_0) = \mathrm{ht}(b_0) = \mathrm{ht}(b_0 \wedge_U b_1)$. If $a_1 <_T a_0^+$, then since f is strictly increasing, $b_1 <_T b_0^+$, so $a_1 <_U a_0$ and $b_1 <_U b_0$. Therefore, $\mathrm{ht}(a_0 \wedge_U a_1) = \mathrm{ht}(a_1) = \mathrm{ht}(b_1) = \mathrm{ht}(b_0 \wedge_U b_1)$.

Finally, assume that a_0^+ and a_1 are incomparable in T. By Lemma 3.3, b_0^+ and b_1 are also incomparable in T. Since U is a simple extension of T, by Lemma 2.5 we have that $a_0 \wedge_U a_1 = a_0^+ \wedge_T a_1$ and $b_0 \wedge_U b_1 = b_0^+ \wedge_T b_1$. By Lemma 3.7, $\mathrm{ht}(a_0^+ \wedge_T a_1) = \mathrm{ht}(b_0^+ \wedge_T b_1)$. So $\mathrm{ht}(a_0 \wedge_U a_1) = \mathrm{ht}(a_0^+ \wedge_T a_1) = \mathrm{ht}(b_0^+ \wedge_T b_1) = \mathrm{ht}(b_0 \wedge_U b_1)$. □

4. Consistency and ρ-Separation

In this section, we introduce and develop some of the main tools we use to define and analyze our forcing, namely, consistency and ρ-separation. These ideas are natural modifications to the finite context of the notions of consistency and separation which were introduced recently by Stejskalová and the first author for the purpose of forcing automorphisms of an ω_1-tree with countable conditions. Roughly speaking, ρ-separation allows configurations of indexed families of automorphisms which are prohibited by the original definition of separation, but only if they occur low enough in the tree according to the function ρ.

Definition 4.1 (Consistency)**.** *Let T be a standard finite tree and let f be a standard function on T. Let $\beta < \alpha$ be ordinals in* $\mathrm{ht}[T]$. *Suppose that $X \subseteq T_\alpha$ and X has unique drop-downs to β. We say that $X \upharpoonright \beta$ and X are f-*consistent *if for all $x, y \in X$, $f(x \upharpoonright \beta) = y \upharpoonright \beta$ iff $f(x) = y$.*

Note that in the above definition, since f is strictly increasing, $f(x) = y$ implies that $f(x \upharpoonright \beta) = y \upharpoonright \beta$. So consistency is equivalent to the upward direction of the definition, namely, that for all $x, y \in X$, if $f(x \upharpoonright \beta) = y \upharpoonright \beta$ then $f(x) = y$.

The proof the following lemma is routine.

Lemma 4.2. *Let T be a standard finite tree and let f be a standard function on T. Let $\beta < \delta < \alpha$ be ordinals in* $\mathrm{ht}[T]$. *Suppose that $X \subseteq T_\alpha$, X has unique drop-downs to β, and $X \upharpoonright \beta$ and X are f-consistent. Then $X \upharpoonright \delta$ has unique drop-downs to β and $X \upharpoonright \beta$ and $X \upharpoonright \delta$ are f-consistent.*

The next lemma follows immediately from Lemma 3.8.

Lemma 4.3. *Let T be a standard finite tree and let f be a standard function on T. Suppose that U is a simple extension of T and g is the downward closure of f in U. If $\alpha \in$* $\mathrm{ht}[U] \setminus \mathrm{ht}[T]$ *is less than* $\max(\mathrm{ht}[T])$ *and β is the least element of* $\mathrm{ht}[T]$ *above α, then U_α and U_β are g-consistent.*

Definition 4.4 (Separation)**.** *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $\alpha \in$* $\mathrm{ht}[T]$ *be greater than* 0 *and let $a_0, \ldots, a_{n-1}$ be distinct elements of T_α. We say that $\{f_\xi : \xi \in A\}$ is* separated on $(a_0, \ldots, a_{n-1})$ *if for all $i < n$, there exists at most one triple (j, m, τ) such that $j < i$, $m \in \{1, -1\}$, $\tau \in A$, and $f_\tau^m(a_i) = a_j$.*

Under the assumptions of the above definition, for $i, j < n$ we sometimes refer to an equation of the form $f_\tau^m(a_i) = a_j$, where $\tau \in A$ and $m \in \{-1, 1\}$, as a *relation between a_i and a_j with respect to $\{f_\xi : \xi \in A\}$*. Note that if $\{f_\xi : \xi \in A\}$ is separated on $(a_0, \ldots, a_{n-1})$, then for any $B \subseteq A$, $\{f_\xi : \xi \in B\}$ is separated on $(a_0, \ldots, a_{n-1})$.

Definition 4.5 (Separation for Sets)**.** *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $\alpha \in$* $\mathrm{ht}[T]$ *be greater than* 0 *and let $X \subseteq T_\alpha$. We say that $\{f_\xi : \xi \in A\}$ is* separated on X *if there exists some injective tuple $\vec{a} = (a_0, \ldots, a_{n-1})$ which lists the elements of X such that $\{f_\xi : \xi \in A\}$ is separated on $\vec{a}$.*

Lemma 4.6. *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Suppose that $\alpha \in$* $\mathrm{ht}[T]$ *is greater than* 0*, $X \subseteq T_\alpha$, and $\{f_\xi : \xi \in A\}$ is separated on X. Then for all $x, y \in X$ and for all pairs (m, τ) and (n, σ) in $\{-1, 1\} \times A$, if $f_\tau^m(x) = y$ and $f_\sigma^n(x) = y$, then $(m, \tau) = (n, \sigma)$.*

Proof. Fix an injective tuple $\vec{a} = (a_0, \ldots, a_{q-1})$ which lists the elements of X so that $\{f_\xi : \xi \in A\}$ is separated on $\vec{a}$. Fix $i, j < q$ so that $a_i = x$ and $a_j = y$. By replacing m and n with $-m$ and $-n$ if necessary, we may assume without loss of generality that $j < i$.

Then $f_\tau^m(a_j) = a_i$ and $f_\sigma^n(a_j) = a_i$, which by separation imply that the triples (i, m, τ) and (i, n, σ) are equal. So $(m, \tau) = (n, \sigma)$. □

Lemma 4.7 (Strong Persistence)**.** *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $0 < \alpha < \beta$ be in $\mathrm{ht}[T]$ and let $X \subseteq T_\alpha$. Assume that $\{f_\xi : \xi \in A\}$ is separated on X. Then $\{f_\xi : \xi \in A\}$ is separated on the set $Y = \{b \in T_\beta : \exists x \in X\ x <_T b\}$.*

Proof. Fix an injective tuple $\vec{a} = (a_0, \ldots, a_{n-1})$ which lists the elements of X so that $\{f_\xi : \xi \in A\}$ is separated on $\vec{a}$. Now let $\vec{b} = (b_0, \ldots, b_{p-1})$ be any injective tuple which lists the elements of Y so that for all $j < i < p$, if j^* and i^* are such that $b_j \restriction \alpha = a_{j^*}$ and $b_i \restriction \alpha = a_{i^*}$, then $j^* \leq i^*$. For each $i < p$, let $i^* < n$ be such that $b_i \restriction \alpha = a_{i^*}$.

Suppose for a contradiction that $i < p$ and there exist distinct triples (j_0, m_0, τ_0) and (j_1, m_1, τ_1) such that for each $k < 2$, $j_k < i$, $m_k \in \{1, -1\}$, $\tau_k \in A$, and $f_{\tau_k}^{m_k}(b_i) = b_{j_k}$. Consider $k < 2$. Then $j_k^* \leq i^*$. Since $f_{\tau_k}^{m_k}$ is strictly increasing, $f_{\tau_k}^{m_k}(a_{i^*}) = a_{j_k^*}$. As f_{τ_k} has no fixed points other than 0, $j_{k^*} < i^*$. Because $\{f_\xi : \xi \in A\}$ is separated on $\vec{a}$, the triples (j_0^*, m_0, τ_0) and (j_1^*, m_1, τ_1) are equal. Hence, $m_0 = m_1$ and $\tau_0 = \tau_1$. So $b_{j_0} = f_{\tau_0}^{m_0}(b_i) = f_{\tau_1}^{m_1}(b_i) = b_{j_1}$. Therefore, $j_0 = j_1$. But then the triples (j_0, m_0, τ_0) and (j_1, m_1, τ_1) are equal, which is a contradiction. □

For the remainder of the article we assume that $\square_{\omega_1}$ holds, and we work with a function ρ whose existence follows from $\square_{\omega_1}$. The function ρ was introduced by Todorčević [Tod87, Section 2]. The basic properties of ρ which we use are as follows:

- ρ is a function with domain ω_2^2 and codomain ω_1;
- $\rho(\alpha, \alpha) = 0$ for all $\alpha < \omega_2$;
- $\rho(\alpha, \beta) = \rho(\beta, \alpha)$ for all $\alpha, \beta < \omega_2$;
- let F be an uncountable family of finite subsets of ω_2 and let $\mu < \omega_1$; then there exist distinct x and y in F such that for all $\tau \in x \setminus y$, $\zeta \in y \setminus x$, and $\gamma \in x \cap y$,
$$\rho(\tau, \zeta) \geq \max\{\rho(\tau, \gamma), \rho(\zeta, \gamma), \mu\}.$$

Note that in last property, if $x \cap y \neq \emptyset$, then for all $\tau \in x \setminus y$ and $\zeta \in y \setminus x$, $\rho(\tau, \zeta) \geq \mu$. We refer to the last bullet point above as the *special property of* ρ. It was proven in [Vel92, Lemma 4.5].

We now introduce our ρ-variation of the concept of separation.

Definition 4.8 (ρ-Separation)**.** *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $\alpha \in \mathrm{ht}[T]$ be greater than 0 and let $a_0, \ldots, a_{n-1}$ be distinct elements of T_α. We say that $\{f_\xi : \xi \in A\}$ is* ρ-separated on *$(a_0, \ldots, a_{n-1})$ if for all $i < n$, if (j_0, m_0, τ_0) and (j_1, m_1, τ_1) are distinct triples satisfying that for each $k < 2$, $j_k < i$, $m_k \in \{-1, 1\}$, $\tau_k \in A$, and $f_{\tau_k}^{m_k}(a_i) = a_{j_k}$, then $j_0 = j_1$ and $\rho(\tau_0, \tau_1) \geq \alpha$.*

Note that separation implies ρ-separation. If $\{f_\xi : \xi \in A\}$ is ρ-separated on the tuple $(a_0, \ldots, a_{n-1})$, then for any $B \subseteq A$, $\{f_\xi : \xi \in B\}$ is ρ-separated on $(a_0, \ldots, a_{n-1})$.

The proof of the next lemma is easy.

Lemma 4.9. *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $\alpha \in \mathrm{ht}[T]$ be greater than 0 and let $a_0, \ldots, a_{n-1}$ be distinct* *elements of T_α. Suppose that for all σ and τ in A, $\rho(\sigma,\tau) < \alpha$. Then $\{f_\xi : \xi \in A\}$ is separated on $(a_0,\dots,a_{n-1})$ iff $\{f_\xi : \xi \in A\}$ is ρ-separated on $(a_0,\dots,a_{n-1})$.*

Definition 4.10 (ρ-Separation for Sets)**.** *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $\alpha \in \mathrm{ht}[T]$ be greater than 0 and let $X \subseteq T_\alpha$. We say that $\{f_\xi : \xi \in A\}$ is* ρ-separated on X *if there exists some injective tuple $\vec{a} = (a_0,\dots,a_{n-1})$ which lists the elements of X such that $\{f_\xi : \xi \in A\}$ is ρ-separated on $\vec{a}$.*

Lemma 4.11. *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $\alpha \in \mathrm{ht}[T]$ be greater than 0 and let $X \subseteq T_\alpha$. Suppose that $\{f_\xi : \xi \in A\}$ is ρ-separated on X. If $Y \subseteq T_\alpha \setminus X$ and for all $\tau \in A$, Y is disjoint from the domain and range of f_τ, then $\{f_\xi : \xi \in A\}$ is ρ-separated on $X \cup Y$.*

Proof. Fix an injective tuple $\vec{a}$ which lists the elements of X so that $\{f_\xi : \xi \in A\}$ is ρ-separated on $\vec{a}$. Let $\vec{b}$ be any injective tuple which lists the elements of Y. Since there are no relations between the elements of Y and $X \cup Y$ with respect to $\{f_\xi : \xi \in A\}$, it easily follows that $\{f_\xi : \xi \in A\}$ is ρ-separated on the concatenation $\vec{a}^\frown\vec{b}$. □

Lemma 4.12 (Downward Persistence)**.** *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $0 < \alpha < \beta$ be in $\mathrm{ht}[T]$. Suppose that $X \subseteq T_\beta$ has unique drop-downs to α and for all $\tau \in A$, $X \restriction \alpha$ and X are f_τ-consistent. If $\{f_\xi : \xi \in A\}$ is ρ-separated on X, then $\{f_\xi : \xi \in A\}$ is ρ-separated on $X \restriction \alpha$.*

Proof. Let $\vec{a} = (a_0,\dots,a_{n-1})$ be an injective tuple which lists X so that $\{f_\xi : \xi \in A\}$ is ρ-separated $\vec{a}$. By unique drop-downs, $\vec{a} \restriction \alpha = (a_0 \restriction \alpha,\dots,a_{n-1} \restriction \alpha)$ is an injective tuple which lists the elements of $X \restriction \alpha$. We claim that $\{f_\xi : \xi \in A\}$ is ρ-separated on $\vec{a} \restriction \alpha$. Suppose that $i < n$ and (j_0, m_0, τ_0) and (j_1, m_1, τ_1) are distinct triples satisfying that for each $k < 2$, $j_k < i$, $m_k \in \{-1, 1\}$, $\tau_k \in A$, and $f_{\tau_k}^{m_k}(a_i \restriction \alpha) = a_{j_k} \restriction \alpha$. By consistency, $f_{\tau_k}^{m_k}(a_i) = a_{j_k}$. As $\{f_\xi : \xi \in A\}$ is ρ-separated on $\vec{a}$, $j_0 = j_1$ and $\rho(\tau_0,\tau_1) \geq \beta > \alpha$. □

Lemma 4.13. *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite set of standard functions on T. Suppose that U is a simple extension of T. For each $\tau \in A$, let g_τ be the downward closure of f_τ in U. Assume that $\alpha \in \mathrm{ht}[U] \setminus \mathrm{ht}[T]$ is less than $\max(\mathrm{ht}[T])$, β is the least element of $\mathrm{ht}[T]$ above α, and $\{f_\xi : \xi \in A\}$ is ρ-separated on T_β. Then $\{g_\tau : \tau \in A\}$ is ρ-separated on U_α. In particular, if $\{f_\xi : \xi \in A\}$ is ρ-separated on T_β for all $\beta \in \mathrm{ht}[T]$, then $\{g_\tau : \tau \in A\}$ is ρ-separated for all $\beta \in \mathrm{ht}[U]$.*

Proof. Immediate from Lemmas 4.3 and 4.12 (Downward Persistence). □

Proposition 4.14 (Characterization of ρ-Separation)**.** *Let T be a standard finite tree and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Let $\alpha \in \mathrm{ht}[T]$ be greater than 0 and let $X \subseteq T_\alpha$. Then $\{f_\xi : \xi \in A\}$ is ρ-separated on X if and only if:*

(1) *for all $x, y \in X$, if (m_0, τ_0) and (m_1, τ_1) are distinct pairs, where for each $k < 2$, $m_k \in \{-1, 1\}$, $\tau_k \in A$, and $f_{\tau_k}^{m_k}(x) = y$, then $\rho(\tau_0, \tau_1) \geq \alpha$;*
(2) *there does not exist a* loop with respect to $\{f_\xi : \xi \in A\}$, *by which we mean a sequence $\langle c_0,\dots,c_{p-1}\rangle$ such that $p \geq 4$, $\langle c_0,\dots,c_{p-2}\rangle$ is injective, $c_0 = c_{p-1}$, and for all $i < p-1$ there exists some $\tau \in A$ and $m \in \{-1, 1\}$ such that $f_\tau^m(c_i) = c_{i+1}$.*

Proof. Suppose that $\{f_\xi : \xi \in A\}$ is ρ-separated on X, and let $\vec{a} = (a_0, \ldots, a_{n-1})$ be an injective tuple which lists X such that $\{f_\xi : \xi \in A\}$ is separated on $\vec{a}$. We prove that (1) and (2) hold.

(1) Suppose that $x, y \in X$ and (m_0, τ_0) and (m_1, τ_1) are distinct pairs satisfying that for each $k < 2$, $m_k \in \{-1, 1\}$, $\tau_k \in A$, and $f_{\tau_k}^{m_k}(x) = y$. We prove that $\rho(\tau_0, \tau_1) \geq \alpha$. Fix j and i less than n such that $x = a_i$ and $y = a_j$. By replacing m_0 and m_1 by $-m_0$ and $-m_1$ if necessary, we may assume without loss of generality that $j < i$. Since the triples (j, m_0, τ_0) and (j, m_1, τ_1) are distinct, it follows by ρ-separation that $\rho(\tau_0, \tau_1) \geq \alpha$.

(2) Suppose for a contradiction that there exists a sequence $\langle c_0, \ldots, c_{p-1} \rangle$ such that $p \geq 4$, $\langle c_0, \ldots, c_{p-2} \rangle$ is injective, $c_0 = c_{p-1}$, and for all $i < p - 1$ there exists some $\tau_i \in A$ and $m_i \in \{-1, 1\}$ such that $f_{\tau_i}^{m_i}(c_i) = c_{i+1}$. For each $i < p$ fix $j_i < n$ such that $c_i = a_{j_i}$. By shifting the sequence if necessary, we may assume without loss of generality that $j_0 = \max\{j_i : i < p - 1\}$. Let $j^* = j_0$. Note that $j^* = j_{p-1}$, $j_1 < j^*$, and $j_{p-2} < j^*$. Then $f_{\tau_0}^{m_0}(a_{j^*}) = a_{j_1}$ and $f_{\tau_{p-2}}^{-m_{p-2}}(a_{j^*}) = a_{j_{p-2}}$. But the triples (j_1, m_0, τ_0) and $(j_{p-2}, -m_{p-2}, \tau_{p-2})$ are distinct and $j_1 \neq j_{p-2}$, which contradicts ρ-separation.

Conversely, assume that (1) and (2) hold and we prove that $\{f_\xi : \xi \in A\}$ is ρ-separated on X. We build by induction an injective tuple $\vec{a} = (a_0, \ldots, a_{n-1})$ which lists the elements of X so that $\{f_\xi : \xi \in A\}$ is ρ-separated on $\vec{a}$. This tuple splits into consecutive segments satisfying:

(a) if c and d are in X and are in distinct segments of $\vec{a}$, then there does not exist $(m, \tau) \in \{-1, 1\} \times A$ such that $f_\tau^m(c) = d$;
(b) for each member b of a segment different from the first element a of that segment, there exists a finite sequence $\langle c_0, \ldots, c_p \rangle$ such that $c_0 = b$, $c_p = a$, and for all $i < p$, c_{i+1} appears earlier in the segment than c_i does and for some $(m, \tau) \in \{-1, 1\} \times A$, $f_\tau^m(c_i) = c_{i+1}$.

For the first member of the first segment, let a_0 be an arbitrary member of X. Assuming that $(a_0, \ldots, a_k)$ has been defined and is part of the first segment, let a_{k+1} be any element of $X \setminus \{a_0, \ldots, a_k\}$ satisfying that there exist $i \leq k$, $\tau \in A$, and $m \in \{-1, 1\}$ such that $f_\tau^m(a_{k+1}) = a_i$. Note that (b) holds for a_{k+1} assuming that it holds for a_i. If there does not exist such an element a_{k+1}, then we move on to the next segment using the same instructions as above, by picking the first element of the next segment arbitrarily and successively choosing new elements related to earlier members of the segment. We continue in this manner defining a sequence of segments, and stop once we have listed all of the members of X. This completes the definition of $\vec{a} = (a_0, \ldots, a_{n-1})$, and properties (a) and (b) clearly hold.

Let us prove that $\{f_\xi : \xi \in A\}$ is ρ-separated on $\vec{a}$. Consider $i < n$, and suppose that (j_0, m_0, τ_0) and (j_1, m_1, τ_1) are distinct triples satisfying that for each $k < 2$, $j_k < i$, $m_k \in \{-1, 1\}$, $\tau_k \in A$, and $f_{\tau_k}^{m_k}(a_i) = a_{j_k}$. We prove that $j_0 = j_1$ and $\rho(\tau_0, \tau_1) \geq \alpha$.

By (a), a_i, a_{j_0}, and a_{j_1} must belong to the same segment of $\vec{a}$. Let d be the first member of this segment. We claim that $j_0 = j_1$. Suppose not. For each $k < 2$, let $\vec{c}^k = (c_0^k, \ldots, c_{p^k}^k)$ be a sequence as described in (b) satisfying that $c_0^k = a_{j_k}$ and $c_{p^k}^k = d$. Since these sequences both end at d, we can fix fix $1 \leq q^0 \leq p^0$ and $1 \leq q^1 \leq p^1$ such that $c_{q^0}^0 = c_{q^1}^1$ but for all $r < q^0$ and $s < q^1$, $c_r^0 \neq c_s^1$. Now the sequence resulting from the concatenation of the four sequences $\langle a_i \rangle$, $\langle c^0_0, \dots, c^0_{q^0} \rangle$, the reverse of $\langle c^1_0, \dots, c^1_{q^1-1} \rangle$, and $\langle a_i \rangle$ is a loop, contradicting (2).

So indeed $j_0 = j_1$. Let $j^* = j_0$. Then the pairs (m_0, τ_0) and (m_1, τ_1) are distinct and for each $k < 2$, $f^{m_k}_{\tau_k}(a_i) = a_j$. By (1), $\rho(\tau_0, \tau_1) \geq \alpha$. □

The proof of the following is very similar to the proof of [KS, Proposition 5.12].

Proposition 4.15 (1-Key Property)**.** *Let T be a standard finite tree which is normal and let $\{f_\xi : \xi \in A\}$ be a finite indexed family of standard functions on T. Assume:*

- $0 < \alpha < \beta$ *are in* $\mathrm{ht}[T]$*;*
- $X \subseteq T_\alpha$*;*
- $\{f_\xi : \xi \in A\}$ *is separated on X;*
- *whenever $f_\tau(x) = y$ holds, where $x, y \in X$ and $\tau \in A$, then* $\mathrm{Succ}_T(x) \subseteq \mathrm{dom}(f_\tau)$ *and* $\mathrm{Succ}_T(y) \subseteq \mathrm{ran}(f_\tau)$.

Then for all $b \in T_\beta$ such that $b \restriction \alpha \in X$, there exists a set $Y \subseteq T_\beta$ with unique drop-downs to α such that $Y \restriction \alpha = X$, $b \in Y$, and for all $\tau \in A$, X and Y are f_τ-consistent.

Proof. Let $\vec{a} = (a_0, \dots, a_{n-1})$ be an injective tuple which lists the elements of X in such a way that $\{f_\xi : \xi \in A\}$ is separated on $\vec{a}$. Fix $\bar{n} < n$ such that $b \restriction \alpha = a_{\bar{n}}$.

We claim that there exists a sequence

$$\langle i_0, (i_1, m_1, \tau_1), \dots, (i_{l-1}, m_{l-1}, \tau_{l-1}) \rangle,$$

for some $l \leq \bar{n} + 1$, such that:

(1) $\bar{n} = i_0 > i_1 > \dots > i_{l-1} \geq 0$;
(2) for all $0 < k < l$, $\tau_k \in A$, $m_k \in \{-1, 1\}$, and $f^{m_k}_{\tau_k}(a_{i_{k-1}}) = a_{i_k}$;
(3) there does not exist a triple (i, m, τ) such that $i < i_{l-1}$, $m \in \{-1, 1\}$, $\tau \in A$, and $f^m_\tau(a_{i_{l-1}}) = a_i$.

We construct the desired sequence by induction. Let $i_0 = \bar{n}$. Now let $k \geq 0$ and assume that we have defined $\langle i_0, (i_1, m_1, \tau_1), \dots, (i_k, m_k, \tau_k) \rangle$ as described in (1) and (2). If there does not exist a triple (i, m, τ) such that $i < i_k$, $m \in \{-1, 1\}$, $\tau \in A$, and $f^m_\tau(a_{i_k}) = a_i$, then let $l = k + 1$ and we are done. Otherwise, fix such a triple (i, m, τ) and let $i_{k+1} = i$, $m_{k+1} = m$, and $\tau_{k+1} = \tau$. This completes the construction. Note that (1) implies that $l \leq \bar{n} + 1$.

Define a tuple $(c_0, \dots, c_{l-1})$ by induction as follows, maintaining that for all $k < l$, $a_{i_k} <_T c_k$. Let $c_0 = b$. Then $a_{i_0} = a_{\bar{n}} <_T b = c_0$. Suppose that $0 < k < l$ and c_{k-1} is defined so that $a_{i_{k-1}} <_T c_{k-1}$. By (2), $f^{m_k}_{\tau_k}(a_{i_{k-1}}) = a_{i_k}$. By the last bullet point in the assumptions of the proposition, c_{k-1} is in the domain of $f^{m_k}_{\tau_k}$. Define $c_k = f^{m_k}_{\tau_k}(c_{k-1})$. Since $f^{m_k}_{\tau_k}$ is strictly increasing, $a_{i_k} <_T c_k$. This completes the definition of $(c_0, \dots, c_{l-1})$.

We now construct the set Y as described in the conclusion of the proposition. By induction on $i < n$ we choose b_i in T_β above a_i, and let $Y = \{b_0, \dots, b_{n-1}\}$. We maintain that for all $k < n$:

(a) for all $\tau \in A$, $(a_0, \dots, a_k)$ and $(b_0, \dots, b_k)$ are f_τ-consistent;
(b) for all $m < l$, if $i_m \leq k$ then $b_{i_m} = c_m$.

Assuming that we are able to define $(b_0, \dots, b_{n-1})$ with these properties, then for all $\tau \in A$, $(a_0, \dots, a_{n-1})$ and $(b_0, \dots, b_{n-1})$ are f_τ-consistent, and $b_{\bar{n}} = b_{i_0} = c_0 = b$, which completes the proof.

For the base case, if $0 \in \{i_0, \dots, i_{l-1}\}$, then clearly $0 = i_{l-1}$, so in this case we let $b_0 = b_{i_{l-1}} = c_{l-1}$. Otherwise, let b_0 be an arbitrary element of T_β above a_0. Clearly, the inductive hypotheses are maintained.

Now let $0 < k < n$ and assume that we have chosen b_i for all $i < k$ so that $(b_0, \dots, b_{k-1})$ satisfies (a) and (b).

Case 1: There does not exist a triple (j, m, τ) such that $j < k$, $m \in \{-1, 1\}$, $\tau \in A$, and $f_\tau^m(a_k) = a_j$. If $k \in \{i_0, \dots, i_{l-1}\}$, then clearly $k = i_{l-1}$, and we let $b_k = b_{i_{l-1}} = c_{l-1}$. So inductive hypothesis (b) holds. Otherwise, choose b_k above a_k arbitrarily. The inductive hypothesis together with the fact that there are no relations between a_k and members of $(a_0, \dots, a_{k-1})$ with respect to $\{f_\xi : \xi \in A\}$ easily imply inductive hypothesis (a).

Case 2: There exists a triple (j, m, σ) such that $j < k$, $m \in \{-1, 1\}$, $\sigma \in A$, and $f_\sigma^m(a_k) = a_j$. By separation, the triple (j, m, σ) is unique. Note that $a_k = f_\sigma^{-m}(a_j)$. By the last bullet point in the assumptions of the proposition, b_j is in the domain of f_σ^{-m}. Define $b_k = f_\sigma^{-m}(b_j)$. Since $a_j <_T b_j$ and f_σ^{-m} is strictly increasing, $a_k <_T b_k$. By the inductive hypothesis and the uniqueness of the triple (j, m, σ), it easily follows that for all $\tau \in A$, $(a_0, \dots, a_k)$ and $(b_0, \dots, b_k)$ are f_τ-consistent. So (a) holds. In the case that $k \in \{i_0, \dots, i_{l-1}\}$, by the uniqueness of the triple (j, m, σ) and the assumption of Case 2, it must be the case that $k = i_{q-1}$ for some q such that $0 < q \leq l-1$, $j = i_q$, $m = m_q$, and $\sigma = \tau_q$. By the induction hypothesis, $b_{i_q} = c_q$, and by the definitions of $b_{i_{q-1}}$ and c_q, $b_{i_{q-1}} = f_{\tau_q}^{-m_q}(b_{i_q}) = f_{\tau_q}^{-m_q}(c_q) = f_{\tau_q}^{-m_q}(f_{\tau_q}^{m_q}(c_{q-1})) = c_{q-1}$. Thus, (b) holds. □

5. The Forcing Poset

We now have the tools at hand to introduce and develop our forcing poset for adding a Suslin tree together with ω_2-many automorphisms of it.

Definition 5.1. *Let $\mathbb{P}$ be the forcing poset consisting of conditions which are pairs (T, F) satisfying:*

1. *T is a standard finite tree;*
2. *F is a function whose domain is a finite subset of ω_2, and for all $\gamma \in \mathrm{dom}(F)$, $F(\gamma)$ is a standard function on T;*
3. *for all $\alpha \in \mathrm{ht}[T]$ greater than 0, $\{F(\tau) : \tau \in \mathrm{dom}(F)\}$ is ρ-separated on T_α.*

Let $(U, G) \leq (T, F)$ if:

- (a) *U extends T;*
- (b) $\mathrm{dom}(F) \subseteq \mathrm{dom}(G)$ *and for all $\gamma \in \mathrm{dom}(F)$, $F(\gamma) \subseteq G(\gamma)$;*
- (c) *suppose that γ and τ are distinct elements of* $\mathrm{dom}(F)$*, x is in* $\mathrm{dom}(G(\gamma)) \cap \mathrm{dom}(G(\tau))$*, and $G(\gamma)(x) = G(\tau)(x)$; then there exists some $z \in T$ such that $x \leq_U z$ and $F(\gamma)(z) = F(\tau)(z)$.*

Going forward, we abbreviate (3) by writing that F is ρ-separated on T_α.

The first component of a condition in $\mathbb{P}$ approximates a tree whose $\mathbb{P}$-name we write as $T^{\dot{G}}$, and the second component approximates a sequence of functions on $T^{\dot{G}}$ whose $\mathbb{P}$-names we write as $F_\tau^{\dot{G}}$ for all $\tau < \omega_2$. We prove in Section 7 that the forcing poset $\mathbb{P}$ is c.c.c. and forces that $T^{\dot{G}}$ is Suslin.

In this section we derive some basic lemmas which we use to prove that $\mathbb{P}$ forces that $T^{\dot{G}}$ is a normal ω-ary ω_1-tree and that $\{F_\tau^{\dot{G}} : \tau < \omega_2\}$ is a strongly almost disjoint family of automorphisms of $T^{\dot{G}}$.

Lemma 5.2. *Suppose that $(T, F) \in \mathbb{P}$, U is a standard finite tree which extends T, and* $\mathrm{ht}[U] \cap (\max(h[T]) + 1) = \mathrm{ht}[T]$. *Then $(U, F) \in \mathbb{P}$ and $(U, F) \leq (T, F)$.*

Proof. By Lemma 3.4, F is a standard function on U. By Lemma 4.11, for all $\alpha \in \mathrm{ht}[U]$, F is ρ-separated on U_α. So $(U, F) \in \mathbb{P}$, and it is simple to check that $(U, F) \leq (T, F)$. □

Lemma 5.3. *Suppose that $(T, F) \in \mathbb{P}$ and U is a standard finite tree which is a simple extension of T. Let $\bar{F}$ be the function with domain equal to* $\mathrm{dom}(F)$ *such that for all $\tau \in \mathrm{dom}(F)$, $\bar{F}(\tau)$ is the downward closure of $F(\tau)$ in U. Then $(U, \bar{F}) \in \mathbb{P}$ and $(U, \bar{F}) \leq (T, F)$.*

Proof. For proving that $(U, \bar{F}) \in \mathbb{P}$, we know that U is a standard finite tree by assumption, for all $\tau \in \mathrm{dom}(\bar{F})$, $\bar{F}(\tau)$ is a standard function on U by Lemma 3.8, and for all $\alpha \in \mathrm{ht}[U]$, $\bar{F}$ is ρ-separated on U_α by Lemma 4.13. For showing that $(U, \bar{F}) \leq (T, F)$, properties (a) and (b) of Definition 5.1 are immediate, and property (c) is easy to check. □

Lemma 5.4. *Let $Z \subseteq \omega_1$ be finite. Then the set of $(U, G) \in \mathbb{P}$ such that $Z \subseteq \mathrm{ht}[U]$ is dense. In fact, for all $(T, F) \in \mathbb{P}$, there exists $(U, G) \leq (T, F)$ such that U is a simple extension of T and* $\mathrm{ht}[U] = \mathrm{ht}[T] \cup Z$.

Proof. Let $(T, F) \in \mathbb{P}$. By Lemma 2.6, we can find a standard finite tree U which is a simple extension of T such that $\mathrm{ht}[U] = \mathrm{ht}[T] \cup Z$. Define G with domain equal to $\mathrm{dom}(F)$ so that for each $\gamma \in \mathrm{dom}(G)$, $G(\gamma)$ is the downward closure of $F(\gamma)$ in U. By Lemma 5.3, $(U, G) \in \mathbb{P}$ and $(U, G) \leq (T, F)$. □

Lemma 5.5. *Let $(T, F) \in \mathbb{P}$ and $x \in T$. Then for any $k > 0$, there exists $(W, H) \leq (T, F)$ such that* $\mathrm{ht}(x) + 1 \in \mathrm{ht}[W]$ *and* $|\mathrm{ISucc}_W(x)| \geq k$.

Proof. By Lemma 5.4, fix $(U, G) \leq (T, F)$ such that U is a simple extension of T and $\mathrm{ht}(x) + 1 \in \mathrm{ht}[U]$. If $|\mathrm{ISucc}_U(x)| \geq k$, then we are done. Otherwise, apply Lemma 2.10 (letting $X = \{x\}$ and $n = k$) to find a standard finite tree W extending U such that $\mathrm{ht}[W] = \mathrm{ht}[U]$, $W \setminus U \subseteq \mathrm{ISucc}_W(x)$, and $|\mathrm{ISucc}_W(x)| = k$. By Lemma 5.2, $(W, G) \in \mathbb{P}$ and $(W, G) \leq (U, G)$. □

Lemma 5.6. *The set of conditions $(U, G) \in \mathbb{P}$ such that U is Hausdorff is dense.*

Proof. Let $(T, F) \in \mathbb{P}$. For each limit ordinal $\delta \in \mathrm{ht}[T]$, fix a successor ordinal δ^- such that $\max(\mathrm{ht}[T] \cap \delta) < \delta^- < \delta$. Let

$$Z = \{\delta^- : \delta \in \mathrm{ht}[T],\ \delta \text{ is a limit ordinal}\}$$

Apply Lemma 5.4 to find $(U, G) \leq (T, F)$ such that U is a simple extension of T and $\mathrm{ht}[U] = \mathrm{ht}[T] \cup Z$. To show that U is Hausdorff, consider distinct x and y in U_δ, where

$\delta \in \mathrm{ht}[U]$ is a limit ordinal. Since $\mathrm{ht}[U] \setminus \mathrm{ht}[T]$ consists of successor ordinals, $\delta \in \mathrm{ht}[T]$, so x and y are in T_δ. As U is a simple extension of T, $x \restriction \delta^- \neq y \restriction \delta^-$. □

Lemma 5.7. *The set of conditions $(U, G) \in \mathbb{P}$ such that U is normal is dense.*

Proof. Let $(T, F) \in \mathbb{P}$. By Lemma 2.9, we can fix a standard finite tree U which is normal such that U extends T and $\mathrm{ht}[U] = \mathrm{ht}[T]$. By Lemma 5.2, $(U, F) \in \mathbb{P}$ and $(U, F) \leq (T, F)$. □

Lemma 5.8. *For any $(T, F) \in \mathbb{P}$, $x \in T$, and countable $\alpha > \mathrm{ht}(x)$, there exists $(W, H) \leq (T, F)$ such that $\alpha \in \mathrm{ht}[W]$ and for some $y \in W_\alpha$, $x <_W y$.*

Proof. By Lemma 5.4, we can fix $(U, G) \leq (T, F)$ such that $\alpha \in \mathrm{ht}[U]$. Now apply Lemma 5.7 to find $(W, H) \leq (U, G)$ such that W is normal. □

Definition 5.9. *For any generic filter G on $\mathbb{P}$, let T^G be the tree with underlying set $\bigcup\{T : \exists F \ (T, F) \in G\}$, and ordered by $x <_{T_G} y$ if there exists some $(T, F) \in G$ such that $x <_T y$. Let $T^{\dot{G}}$ be a $\mathbb{P}$-name for this object.*

Proposition 5.10. *The forcing poset $\mathbb{P}$ forces:*

- *$T^{\dot{G}}$ is a tree with height ω_1^V and countable levels;*
- *$T^{\dot{G}}$ is normal and ω-ary.*

By normal, we mean that the tree has a root, every element has at least two immediate successors, it is Hausdorff, and every element has elements above it at every higher level. By ω-ary, we mean that every element of the tree has ω-many immediate successors.

Proof. Let G be a generic filter on $\mathbb{P}$. It is easy to prove that $(T^G, <_{T^G})$ is a tree (the well-foundedness follows from the fact that $x <_{T^G} y$ implies that $\mathrm{ht}(x) < \mathrm{ht}(y)$). Lemma 5.4 implies that the height function on T^G coincides with the height function ht that we defined on countable ordinals. It then follows that the levels of T^G are countable and T^G has height ω_1^V. For being ω-ary and normal, clearly 0 is the root of T^G, Lemma 5.5 implies that T^G is ω-ary, Lemma 5.6 implies that T^G is Hausdorff, and Lemma 5.8 completes the proof. □

Corollary 5.11. *Assuming that $\mathbb{P}$ preserves ω_1, $\mathbb{P}$ forces that $T^{\dot{G}}$ is a normal ω-ary ω_1-tree.*

We prove in Section 7 that $\mathbb{P}$ is c.c.c., and hence $\mathbb{P}$ preserves ω_1. We also prove in Section 7 that $\mathbb{P}$ forces that $T^{\dot{G}}$ is Suslin.

Lemma 5.12. *Suppose that $(T, F) \in \mathbb{P}$ and $\sigma < \omega_2$. Then there exists $(U, G) \leq (T, F)$ such that $\sigma \in \mathrm{dom}(G)$.*

Proof. If $\sigma \in \mathrm{dom}(F)$, then we are done. Otherwise, define G with domain equal to $\mathrm{dom}(F) \cup \{\sigma\}$, where $G \restriction \mathrm{dom}(F) = F$ and $G(\sigma) = \emptyset$. It is simple to check that $(T, G) \in \mathbb{P}$ and $(T, G) \leq (T, F)$. □

Lemma 5.13 (Augmentation)**.** *Suppose that $(T, F) \in \mathbb{P}$, $\sigma < \omega_2$, and $x \in T$. Then there exists $(U, G) \leq (T, F)$ such that $\sigma \in \mathrm{dom}(G)$ and x is in the domain and range of $G(\sigma)$.*

Proof. By Lemma 5.12, without loss of generality we may assume that $\sigma \in \mathrm{dom}(F)$. We prove the statement about x being in the domain by induction on the height of x; the proof for the range is similar. For the base case, if x has height 0, then $x = 0$. So we can extend (T, F) as required by mapping 0 to 0. Now suppose that x has height $\alpha > 0$. If $x \in \mathrm{dom}(F(\sigma))$ then we are done, so assume not. Let x^- be the largest member of $\{z \in T : z <_T x\}$. By the inductive hypothesis, we may assume without loss of generality that $x^- \in \mathrm{dom}(F(\sigma))$. Extend T to U by adding a single element z which is not in T above $F(\sigma)(x^-)$ with height α. Define G with the same domain as F with the only change being that $G(\sigma)(x) = z$.

To prove that (U, G) is a condition, we only show that G is ρ-separated on U_α since the other properties are clear. Let $\vec{a}$ be an injective tuple which lists T_α so that F is ρ-separated on $\vec{a}$. Since the only relation which z has with members of $\vec{a}$ is the equation $G(\sigma)^{-1}(z) = x$, clearly G is ρ-separated on $\vec{a}^\frown z$. Hence, $(U, G) \in \mathbb{P}$, and easily $(U, G) \leq (T, F)$. □

Definition 5.14. *For any generic filter G on $\mathbb{P}$ and $\tau < \omega_2$, define*

$$F_\tau^G = \bigcup\{F(\tau) : \exists T\ (T, F) \in G \text{ and } \tau \in \mathrm{dom}(F)\}.$$

Let $F_\tau^{\dot{G}}$ be a $\mathbb{P}$-name for this object.

Proposition 5.15. *The forcing poset $\mathbb{P}$ forces that for all $\tau < \omega_2$, $F_\tau^{\dot{G}}$ is an automorphism of $T^{\dot{G}}$.*

Proof. A straightforward argument shows that $F_\tau^{\dot{G}}$ is forced to be a function, and by Lemma 5.13, it is total and surjective. Whenever $(T, F) \in \mathbb{P}$ and $\tau \in \mathrm{dom}(F)$, $F(\tau)$ is injective and strictly increasing. It easily follows that $F_\tau^{\dot{G}}$ is forced to be injective and strictly increasing. So $F_\tau^{\dot{G}}$ is forced to be a strictly increasing bijection, and hence an automorphism. □

Proposition 5.16. *The forcing poset $\mathbb{P}$ forces that for all $\gamma < \tau < \omega_2$, $F_\gamma^{\dot{G}}$ and $F_\tau^{\dot{G}}$ are strongly almost disjoint subsets of $T^{\dot{G}} \otimes T^{\dot{G}}$.*

Proof. Let G be a generic filter on $\mathbb{P}$. By Lemma 5.12, we can fix a condition $(T, F) \in G$ such that γ and τ are in $\mathrm{dom}(F)$. Suppose that $F_\gamma^G(x) = y$ and $F_\tau^G(x) = y$. Then clearly there exists some $(W, H) \in G$ such that $(W, H) \leq (T, F)$, $H(\gamma)(x) = y$, and $H(\tau)(x) = y$. By the definition of the ordering of $\mathbb{P}$, there exist z and z^* in T such that $x \leq_W z$, $F(\gamma)(z) = z^*$, and $F(\tau)(z) = z^*$. Hence, in the tree $T^G \otimes T^G$, $(x, y) \leq (z, z^*)$. It follows that $F_\gamma^G \cap F_\tau^G$ is a subset of the downward closure of the finite set $\{(a, b) \in T^G \otimes T^G : F(\gamma)(a) = b\}$, and therefore is a finite union of countable chains. □

In Section 7 we prove that $\mathbb{P}$ is c.c.c. and forces that $T^{\dot{G}}$ is Suslin. These two facts combined with Corollary 5.11 and Propositions 5.15 and 5.16 complete the proof of the main theorem.

6. Making the Functions Bijective

In order to prove that $\mathbb{P}$ forces that the generic tree $T^{\dot{G}}$ is Suslin, we need to apply Proposition 4.15 (1-Key Property). The main challenge in doing this is to construct a condition which satisfies the assumption given in the fourth bullet point of that proposition. Namely, we need to extend a condition so that some of its functions which are separated on a subset of some level of its tree are total and surjective above that subset. In this section we achieve this goal.

Lemma 6.1. *Suppose that $(T, F) \in \mathbb{P}$, $\alpha \in \mathrm{ht}[T] \cap \max(\mathrm{ht}[T])$, $X \subseteq T_\alpha$, and n is a positive natural number such that every element of X has at most n-many immediate successors. Then there exists U such that:*

1. *$(U, F) \in \mathbb{P}$ and $(U, F) \leq (T, F)$;*
2. *$\mathrm{ht}[T] = \mathrm{ht}[U]$;*
3. *$U \setminus T \subseteq \bigcup\{\mathrm{ISucc}_U(x) : x \in X\}$;*
4. *every element of X has exactly n-many immediate successors in U.*

Proof. Apply Lemma 2.10 to find a standard finite tree U which extends T and satisfies (2), (3), and (4). By Lemma 5.2, $(U, F) \in \mathbb{P}$ and $(U, F) \leq (T, F)$. □

Proposition 6.2. *Suppose that $(T, F) \in \mathbb{P}$, $\alpha \in \mathrm{ht}[T] \cap \max(\mathrm{ht}[T])$ is positive, $\beta = \min(\mathrm{ht}[T] \setminus (\alpha + 1))$, $X \subseteq T_\alpha$ is non-empty, and $A \subseteq \mathrm{dom}(F)$. Assume that $\{F(\tau) : \tau \in A\}$ is separated on X. Then there exists $(U, G) \in \mathbb{P}$ satisfying:*

- *$(U, G) \leq (T, F)$;*
- *$\mathrm{ht}[T] = \mathrm{ht}[U]$ and $\mathrm{dom}(F) = \mathrm{dom}(G)$;*
- *$U \setminus T \subseteq \bigcup\{\mathrm{ISucc}_U(x) : x \in X\}$;*
- *for all $x \in X$, $\mathrm{ISucc}_U(x)$ is non-empty;*
- *for all $\tau \in \mathrm{dom}(G)$ and for all $z \in \mathrm{dom}(G(\tau)) \setminus \mathrm{dom}(F(\tau))$, both z and $G(\tau)(z)$ are in $\bigcup\{\mathrm{ISucc}_U(x) : x \in X\}$;*
- *$\{G(\tau) : \tau \in A\}$ is separated on $\bigcup\{\mathrm{ISucc}_U(x) : x \in X\}$;*
- *for all $\tau \in A$ and for all $x, y \in X$, if $x \in \mathrm{dom}(G(\tau))$ and $G(\tau)(x) = y$, then $\mathrm{ISucc}_U(x) \subseteq \mathrm{dom}(G(\tau))$ and $\mathrm{ISucc}_U(y) \subseteq \mathrm{ran}(G(\tau))$.*

Proof. Let $q = |X|$. Fix an injective tuple $(a_0, \ldots, a_{q-1})$ which lists the elements of X so that $\{F(\tau) : \tau \in A\}$ is separated on $(a_0, \ldots, a_{q-1})$. Choose a natural number $p > 0$ such that every element of X has at most p-many immediate successors. Applying Lemma 6.1, fix U such that $(U, F) \in \mathbb{P}$, $(U, F) \leq (T, F)$, $\mathrm{ht}[T] = \mathrm{ht}[U]$, $U \setminus T \subseteq \bigcup\{\mathrm{ISucc}_U(x) : x \in X\}$, and every element of X has exactly pq-many immediate successors in U. Let $Y = \bigcup\{\mathrm{ISucc}_U(x) : x \in X\}$. For each $i < q$, fix a partition $\{X^i_0, \ldots, X^i_{q-1}\}$ of $\mathrm{ISucc}_U(a_i)$ into disjoint sets each of size p and satisfying that $\mathrm{ISucc}_T(a_i) \subseteq X^i_i$.

We define a function G with domain equal to $\mathrm{dom}(F)$ satisfying the following properties for each $\tau \in \mathrm{dom}(F)$:

(I) $G(\tau)$ is a standard function on U and $F(\tau) \subseteq G(\tau)$;
(II) if $\tau \notin A$, then $G(\tau) = F(\tau)$;
(III) if $\tau \in A$, $G(\tau)(x) = y$, and $x \notin \mathrm{dom}(F(\tau))$, then x and y are in Y and $F(\tau)(x \restriction \alpha) = y \restriction \alpha$.

Let us see what conclusions can be drawn from (I)–(III). Note that for all $\tau \in \mathrm{dom}(F)$ and for all $\delta \in \mathrm{ht}[U]$ different from β, $G(\tau) \restriction U_\delta = F(\tau) \restriction U_\delta$. In particular, $\{G(\tau) : \tau \in A\}$ is separated on $(a_0, \ldots, a_{q-1})$. Concerning showing that $(U, G) \in \mathbb{P}$: Definition 5.1(1,2) are clear, and Definition 5.1(3) holds provided that G is ρ-separated on U_β. For showing that $(U, G) \leq (U, F)$, Definition 5.1(a,b) are clear, so it suffices to prove Definition 5.1(c). Assume for a moment that $(U, G) \in \mathbb{P}$ and $(U, G) \leq (U, F)$, and let us review the conclusions of the proposition. The first five bullet points of the proposition are clear, and the sixth follows from the fact that $\{G(\tau) : \tau \in A\}$ is separated on X and Lemma 4.7 (Strong Persistence). So for verifying the conclusions of the proposition, it suffices to prove the seventh bullet point.

To summarize, in order to complete the proof of the proposition, it suffices to define a function G with domain equal to $\mathrm{dom}(F)$ which satisfies (I), (II), and (III) above for all $\tau \in \mathrm{dom}(F)$, and also has the following properties:

(IV) G is ρ-separated on U_β;
(V) for all distinct γ and τ in $\mathrm{dom}(F)$, if $x \in \mathrm{dom}(G(\gamma)) \cap \mathrm{dom}(G(\tau))$ and $G(\gamma)(x) = G(\tau)(x)$, then there exists some $z \in U$ such that $x \leq_U z$ and $F(\gamma)(z) = F(\tau)(z)$;
(VI) for all $\tau \in A$ and for all $x, y \in X$, if $x \in \mathrm{dom}(G(\tau))$ and $G(\tau)(x) = y$, then $\mathrm{ISucc}_U(x) \subseteq \mathrm{dom}(G(\tau))$ and $\mathrm{ISucc}_U(y) \subseteq \mathrm{ran}(G(\tau))$.

Let us now proceed with the definition of G. For all $\tau \in \mathrm{dom}(F) \setminus A$, define $G(\tau) = F(\tau)$. So (II) is satisfied. Now consider $\tau \in A$. Define $G(\tau) \restriction \mathrm{dom}(F(\tau)) = F(\tau)$. Consider $x \in U \setminus \mathrm{dom}(F(\tau))$. We let x be in the domain of $G(\tau)$ if and only if $x \restriction \alpha \in X \cap \mathrm{dom}(F(\tau))$ and $F(\tau)(x \restriction \alpha) \in X$. In that case, we define $G(\tau)(x)$ to be an element of $\mathrm{ISucc}_U(F(\tau)(x \restriction \alpha))$, as described below. In other words, for any $i, j < q$ such that $F(\tau)(a_i) = a_j$, we define $G(\tau)$ on $\mathrm{ISucc}_U(a_i)$ so that it is a bijection between $\mathrm{ISucc}_U(a_i)$ and $\mathrm{ISucc}_U(a_j)$. Let us assume for a moment that we succeed in defining $G(\tau)$ in this manner. It is routine to verify that $G(\tau)$ is a standard function on U, so (I) holds. Also, (III) and (VI) are clear.

So let $i, j < q$ be given such that $F(\tau)(a_i) = a_j$. Since $F(\tau)$ is a standard function, $i \neq j$. Recall that $\{X^i_0, \dots, X^i_{q-1}\}$ is a partition of $\mathrm{ISucc}_U(a_i)$ into disjoint sets each of size p and satisfying that

$$\mathrm{dom}(F(\tau)) \cap \mathrm{ISucc}_U(a_i) \subseteq \mathrm{ISucc}_T(a_i) \subseteq X^i_i.$$

Similarly, $\{X^j_0, \dots, X^j_{q-1}\}$ is a partition of $\mathrm{ISucc}_U(a_j)$ into disjoint sets each of size p and satisfying that

$$\mathrm{ran}(F(\tau)) \cap \mathrm{ISucc}_U(a_j) \subseteq \mathrm{ISucc}_T(a_j) \subseteq X^j_j.$$

We define $G(\tau)$ extending $F(\tau)$ which maps $\mathrm{ISucc}_U(a_i)$ bijectively onto $\mathrm{ISucc}_U(a_j)$ and has the following properties:

(a) for all $k \in q \setminus \{i, j\}$, $G(\tau)[X^i_k] = X^j_k$;
(b) $G(\tau)[X^i_i \setminus \mathrm{dom}(F(\tau))] \subseteq X^j_i$;
(c) $G(\tau)^{-1}[X^j_j \setminus \mathrm{dom}(F(\tau)^{-1})] \subseteq X^i_j$.

Let us see how we can arrange this. By the last paragraph, since $G(\tau) \restriction \mathrm{dom}(F(\tau)) = F(\tau)$, we have specified $G(\tau)(x)$ for those x which are in the set $\mathrm{dom}(F(\tau)) \cap \mathrm{ISucc}_U(a_i)$, and hence are in X^i_i, and the values $G(\tau)(x)$ for such x are in X^j_j. For each $k < q$ different from i and j, X^i_k and X^j_k have the same size p, so we can easily arrange that (a) holds. This defines $G(\tau)$ on $\mathrm{ISucc}_U(a_i) \setminus (X^i_i \cup X^i_j)$ and on $\mathrm{dom}(F(\tau)) \cap X^i_i$. We have not defined $G(\tau)(x)$ yet for any $x \in X^i_j$. Also, the values of $G(\tau)$ which we have defined so far are not in X_i^j. Define $G(\tau)$ on $X_i^i \setminus \mathrm{dom}(F(\tau))$ so that it maps injectively into X_i^j. This completes the definition of $G(\tau)$ on X_i^i and it satisfies (b). It remains to define $G(\tau)$ on X_j^i so that (c) is satisfied. We have not defined any values of $G(\tau)$ on X_j^i yet, so we can do so in such a way that $G(\tau)^{-1}$ maps $X_j^j \setminus \mathrm{ran}(F(\tau))$ into X_j^i. Hence, (c) is satisfied. Now extend the definition of $G(\tau)$ injectively to the rest of X_j^i, mapping to values not already taken (the remaining values of $G(\tau)$ on X_j^i besides those specified above must be in X_i^j, since all of the other values are already taken, although that fact does not matter for us).

This completes the definition of G. It remains to prove (IV) and (V).

(V) Let γ and τ be distinct elements of $\mathrm{dom}(F)$, and assume that $x \in \mathrm{dom}(G(\gamma)) \cap \mathrm{dom}(G(\tau))$ and $G(\gamma)(x) = G(\tau)(x)$. We prove that there exists some $z \in U$ such that $x \leq_U z$ and $F(\gamma)(z) = F(\tau)(z)$. If $x \in \mathrm{dom}(F(\gamma)) \cap \mathrm{dom}(F(\tau))$, then we are done by letting $z = x$. Otherwise, we can assume without loss of generality that $x \notin \mathrm{dom}(F(\gamma))$. Then by (II) and (III), $\gamma \in A$ and both x and $G(\gamma)(x)$ are in Y. Fix $i, j < q$ such that $x \in \mathrm{ISucc}_U(a_i)$ and $G(\gamma)(x) \in \mathrm{ISucc}_U(a_j)$.

If $\tau \in A$, then by Lemma 4.6 the equation $G(\tau)(x) = G(\gamma)(x)$ contradicts the fact that $\{G(\xi) : \xi \in A\}$ is separated on Y. So $\tau \notin A$. By (II), $G(\tau) = F(\tau)$, so $G(\gamma)(x) = F(\tau)(x)$. Consequently, x and $G(\gamma)(x)$ are in T. By the choice of the partitions, $x \in X_i^i$ and $G(\gamma)(x) \in X_j^j$. Since $x \notin \mathrm{dom}(F(\gamma))$, (b) implies that $G(\gamma)(x)$ is in X_i^j, which is a contradiction since X_j^j and X_i^j are disjoint. This completes the proof of (V).

(IV) We now begin the proof that G is ρ-separated on U_β. We need the following three claims.

<u>Claim 1:</u> Let $i, j < q$, $x \in \mathrm{ISucc}_T(a_i)$, $y \in \mathrm{ISucc}_U(a_j) \setminus T$, $\tau \in A$, and $m \in \{-1, 1\}$. Suppose that $G(\tau)^m(x) = y$. Then $y \in X_i^j$.

Proof. Since $y \notin T$, $x \notin \mathrm{dom}(F(\tau)^m)$. On the other hand, $x \in T$, so $x \in X_i^i \setminus \mathrm{dom}(F(\tau)^m)$. By (b) and (c), $y \in X_i^j$. □

<u>Claim 2:</u> Let $i, j, k < q$ and let τ and σ be in $\mathrm{dom}(F)$. Suppose that $x, y, z \in U$ are distinct, $x \in \mathrm{ISucc}_T(a_i)$, $y \in \mathrm{ISucc}_U(a_j) \setminus T$, $z \in \mathrm{ISucc}_U(a_k)$, $m, n \in \{-1, 1\}$, $G(\tau)^m(x) = y$, and $G(\sigma)^n(y) = z$. Then $z \notin T$.

Proof. Since $y \notin T$, $y \notin \mathrm{ran}(F(\tau)^m)$ and $y \notin \mathrm{dom}(F(\sigma)^n)$. By (II), it follows that $\tau \in A$ and $\sigma \in A$. Applying Claim 1 for x and y, we get that $y \in X_i^j$. Suppose for a contradiction that $z \in T$. Applying Claim 1 to z and y and the equation $G(\sigma)^{-n}(z) = y$, we get that $y \in X_k^j$. Consequently, $i = k$. Since $G(\tau)^{-m}(y) = x$ and $G(\sigma)^n(y) = z$, it follows that $F(\tau)^{-m}(a_j) = a_i$ and $F(\sigma)^n(a_j) = a_i$. As $\{F(\xi) : \xi \in A\}$ is separated on X, $\tau = \sigma$ and $-m = n$. But then $x = z$, which is a contradiction. □

<u>Claim 3:</u> There do not exist tuples $(b_0, \ldots, b_{l-1})$, $(m_0, \ldots, m_{l-2})$, and $(\tau_0, \ldots, \tau_{l-2})$, for some natural number $l \geq 3$, satisfying:

- for all $j < l$, $b_j \in Y$;
- $m_0, \ldots, m_{l-2}$ are in $\{-1, 1\}$ and $\tau_0, \ldots, \tau_{l-2}$ are in A;
- $G(\tau_j)^{m_j}(b_j) = b_{j+1}$ for all $j < l - 1$;
- $b_0 \in T$, $b_j \notin T$ for all $0 < j < l - 1$, and $b_{l-1} \in T$;
- $b_0, \ldots, b_{l-2}$ are all distinct;

- b_0, b_1, and b_2 are distinct.

Proof. Suppose for a contradiction that such tuples exist. Fix $i_0, \ldots, i_{l-1} < q$ such that $b_j \in \mathrm{ISucc}_U(a_{i_j})$ for all $j < l$. Note that $F(\tau_j)^{m_j}(a_{i_j}) = a_{i_{j+1}}$ for all $j < l-1$. By Claim 2 applied to b_0, b_1, and b_2, we can conclude that b_2 is not in T. Since $b_{l-1} \in T$, it follows that $l \geq 4$.

We claim that for all $j < k < l-1$, $i_j \neq i_k$. Otherwise, let $k < l-1$ be least such that for some $j < k$, $i_j = i_k$. Since $F(\tau_{k-1})$ has no fixed points in X and $F(\tau_{k-1})^{m_{k-1}}(a_{i_{k-1}}) = a_{i_k}$, $j < k-1$. If $j = k-2$, then we would have that $F(\tau_j)^{-m_j}(a_{i_{j+1}}) = a_{i_j}$ and

$$F(\tau_{j+1})^{m_{j+1}}(a_{i_{j+1}}) = a_{i_{j+2}} = a_{i_k} = a_{i_j},$$

which implies by Lemma 4.6 that $\tau_j = \tau_{j+1}$ and $-m_j = m_{j+1}$. So $G(\tau_j)^{-m_j}(b_{j+1}) = b_j$ would be equal to $G(\tau_{j+1})^{m_{j+1}}(b_{j+1}) = b_{j+2} = b_k$, which contradicts that $b_j \neq b_k$. Hence, $j < k-2$. But then $\langle a_{i_j}, a_{i_{j+1}}, \ldots, a_{i_k} \rangle$ is a loop with respect to $\{F(\xi) : \xi \in A\}$. By Proposition 4.14 (Characterization of ρ-Separation), $\{F(\xi) : \xi \in A\}$ is not ρ-separated on X, and hence is not separated on X, which is a contradiction.

Since $b_0 \in T$, $b_0 \in X_{i_0}^{i_0}$. But $b_0 \notin \mathrm{dom}(F(\tau_0)^{m_0})$, for otherwise b_1 would be in T. By (b) and (c) it follows that $b_1 \in X_{i_0}^{i_1}$. Using the fact that $i_0, i_1, \ldots, i_{l-2}$ are all distinct, it follows by induction using (a) that for all $k < l-1$, $b_k \in X_{i_0}^{i_k}$. In particular, $b_{l-2} \in X_{i_0}^{i_{l-2}}$. By Claim 1 (letting $x = b_{l-1}$ and $y = b_{l-2}$), $b_{l-2} \in X_{i_{l-1}}^{i_{l-2}}$. It follows that $i_{l-1} = i_0$. So $a_{i_{l-1}} = a_{i_0}$. Since $a_0, \ldots, a_{i_{l-2}}$ are distinct and $l \geq 4$, $\langle a_{i_0}, \ldots, a_{i_{l-2}}, a_{i_{l-1}} \rangle$ is a loop with respect to $\{F(\xi) : \xi \in A\}$. By Proposition 4.14 (Characterization of ρ-Separation), $\{F(\xi) : \xi \in A\}$ is not ρ-separated on X, and hence is not separated on X, which is a contradiction. □

Now we are ready to prove that G is ρ-separated on U_β. We verify properties (1) and (2) of Proposition 4.14 (Characterization of ρ-Separation). (1) Since $(U, F) \in \mathbb{P}$, F is ρ-separated on U_β. Suppose that $c, d \in U_\beta$, (m, τ) and (n, σ) are distinct elements of $\{-1, 1\} \times \mathrm{dom}(G)$, and $G(\tau)^m(c) = d$ and $G(\sigma)^n(c) = d$. We show that $\rho(\tau, \sigma) \geq \beta$. If $c \in \mathrm{dom}(F(\tau)^m)$ and $c \in \mathrm{dom}(F(\sigma)^n)$, then $F(\tau)^m(c) = d$ and $F(\sigma)^n(c) = d$, so $\rho(\tau, \sigma) \geq \beta$ since F is ρ-separated on U_β. Otherwise, without loss of generality $c \notin \mathrm{dom}(F(\tau)^m)$. By (II) and (III), $\tau \in A$ and there are distinct $i, j < q$ such that $c \in \mathrm{ISucc}_U(a_i)$ and $d \in \mathrm{ISucc}_U(a_j)$. If $\sigma \in A$, then by Lemma 4.6 we have a contradiction to the fact that $\{G(\xi) : \xi \in A\}$ is separated on Y. So $\sigma \notin A$. Hence, $F(\sigma)^n(c) = d$. So c and d are both in T, and therefore $c \in X_i^i$ and $d \in X_j^j$. On the other hand, $c \notin \mathrm{dom}(F(\tau)^m)$, so by (b) and (c), $G(\tau)^m(c) = d$ is in X_i^j, which contradicts the fact that X_j^j and X_i^j are disjoint.

(2) Suppose for a contradiction that there exists a loop $\langle b_0, \ldots, b_{k-1} \rangle$ in U_β with respect to G. So $k \geq 4$, there exist $\tau_0, \ldots, \tau_{k-2}$ in $\mathrm{dom}(G)$ and $m_0, \ldots, m_{k-2}$ in $\{-1, 1\}$ such that $b_0 = b_{k-1}$, $\langle b_0, \ldots, b_{k-2} \rangle$ is injective, and for all $i < k-1$, $G(\tau_i)^{m_i}(b_i) = b_{i+1}$. If for all $i < k-1$, $b_i \in \mathrm{dom}(F(\tau_i)^{m_i})$, then we have a contradiction to the fact that F is ρ-separated on U_β. If for all $i < k-1$, $\tau_i \in A$ and $b_i \in Y$, then we have a contradiction to the fact that $\{G(\xi) : \xi \in A\}$ is separated on Y.

So we may assume that (i) for some $i < k-1$, $b_i \notin \mathrm{dom}(F(\tau_i)^{m_i})$, and (ii) for some $j < k-1$, either $\tau_j \notin A$ or $b_j \notin Y$. Note that it follows from (i), (II), and (III) that b_i

and b_{i+1} are in Y. For (ii), in either case $b_j \in T$ (namely, use (II) if $\tau_j \notin A$ and the fact that $U \setminus T \subseteq Y$ if $b_j \notin Y$). We claim that in (i), either b_i or b_{i+1} is not in T. If $b_i \notin T$, then we are done, so assume that $b_i \in T$. Fix $s, t < q$ such that $b_i \in \mathrm{ISucc}_U(a_s)$ and $b_{i+1} \in \mathrm{ISucc}_U(a_t)$. Then $s \neq t$. Since $b_i \in T$, $b_i \in X_s^s \setminus \mathrm{dom}(F(\tau_i)^{m_i})$. By (b) and (c), $b_{i+1} \in X_s^t$, and hence $b_{i+1} \notin T$ since otherwise b_{i+1} would be in X_t^t.

In conclusion, some member of the loop is not in T and some member is in T. Obviously, this implies that there are adjacent members of the loop where one is in T and the other is not in T. By shifting the loop if necessary, we may assume without loss of generality that $b_0 \in T$ and $b_1 \notin T$. Then $b_{k-1} = b_0$ is in T. Let $l \leq k$ be the least natural number greater than 1 such that $b_{l-1} \in T$. Since $b_1 \notin T$, by (II) and (III) it follows that $\tau_0 \in A$ and $b_0 \in Y$. Similarly, for all $0 < s < l-1$, $b_s \notin T$ implies by (II) and (III) that $\tau_s \in A$ and both b_s and b_{s+1} are in Y. In particular, $b_{l-1} \in Y$. Since $k \geq 4$, b_0, b_1, and b_2 are all different. As $b_1 \notin T$, $l \geq 3$. Using this information, it is easy to check that $(b_0, \ldots, b_{l-1})$, $(\tau_0, \ldots, \tau_{l-2})$, and $(m_0, \ldots, m_{l-2})$ satisfy the description of the tuples which Claim 3 states does not exist, which is a contradiction. □

Corollary 6.3. *Suppose that $(T, F) \in \mathbb{P}$, $\alpha \in \mathrm{ht}[T]$ is positive, $X \subseteq T_\alpha$ is non-empty, and $A \subseteq \mathrm{dom}(F)$. Assume that $\{F(\tau) : \tau \in A\}$ is separated on X. Then there exists $(U, G) \in \mathbb{P}$ satisfying:*

- $(U, G) \leq (T, F)$;
- $\mathrm{ht}[T] = \mathrm{ht}[U]$ *and* $\mathrm{dom}(F) = \mathrm{dom}(G)$;
- $U \setminus T \subseteq \bigcup\{\mathrm{Succ}_U(x) : x \in X\}$;
- *for all* $\tau \in \mathrm{dom}(G)$ *and for all* $z \in \mathrm{dom}(G(\tau)) \setminus \mathrm{dom}(F(\tau))$, *both* z *and* $G(\tau)(z)$ *are in* $\bigcup\{\mathrm{Succ}_U(x) : x \in X\}$;
- *for all* $\tau \in A$ *and for all* $x, y \in X$, *if* $x \in \mathrm{dom}(G(\tau))$ *and* $G(\tau)(x) = y$, *then* $\mathrm{Succ}_U(x) \subseteq \mathrm{dom}(G(\tau))$ *and* $\mathrm{Succ}_U(y) \subseteq \mathrm{ran}(G(\tau))$.

Proof. By induction on the ordinals in $\mathrm{ht}[T] \setminus (\alpha+1)$, we can build the desired condition in finitely many steps, where at each step we use Proposition 6.2 to go up one more level. □

Corollary 6.4. *Suppose that $(T, F) \in \mathbb{P}$, $\alpha \in \mathrm{ht}[T]$ is positive, $X \subseteq T_\alpha$ is non-empty, and $A \subseteq \mathrm{dom}(F)$. Assume that $\{F(\tau) : \tau \in A\}$ is separated on X. Let $\beta = \max(\mathrm{ht}[T])$ and let $b \in T_\beta$ be such that $b \upharpoonright \alpha \in X$. Then there exists $(U, G) \in \mathbb{P}$ and $Y \subseteq U_\beta$ satisfying:*

(1) $(U, G) \leq (T, F)$;
(2) $\mathrm{ht}[T] = \mathrm{ht}[U]$ *and* $\mathrm{dom}(F) = \mathrm{dom}(G)$;
(3) $U \setminus T \subseteq \bigcup\{\mathrm{Succ}_U(x) : x \in X\}$;
(4) *for all* $\tau \in \mathrm{dom}(G)$ *and for all* $z \in \mathrm{dom}(G(\tau)) \setminus \mathrm{dom}(F(\tau))$, *both* z *and* $G(\tau)(z)$ *are in* $\bigcup\{\mathrm{Succ}_U(x) : x \in X\}$;
(5) *Y has unique drop-downs to α, $Y \upharpoonright \alpha = X$, and $b \in Y$;*
(6) *for all $\tau \in A$, X and Y are $G(\tau)$-consistent.*

Proof. Immediate from Proposition 4.15 (1-Key Property) and Corollary 6.3. □

7. The Generic Tree is Suslin

We now complete the proof of the main theorem by showing that $\mathbb{P}$ is c.c.c. and forces that $T^{\dot{G}}$ is Suslin. These facts will follow from the next theorem (see Corollaries 7.2 and 7.3).

Theorem 7.1. *Suppose that $\langle (T^\alpha, F^\alpha) : \alpha < \omega_1 \rangle$ is a sequence of conditions in $\mathbb{P}$ and for each $\alpha < \omega_1$, $x^\alpha \in T^\alpha \setminus \alpha$. Then there exist $\alpha < \beta < \omega_1$ and a condition (W, H) such that (W, H) extends (T^α, F^α) and (T^β, F^β) and $x^\alpha <_W x^\beta$.*

Proof. By extending further if necessary using Lemmas 5.4, 5.7, and 5.12, we may assume that for all $\alpha < \omega_1$, $\alpha \in \mathrm{ht}[T^\alpha]$, T^α is normal, and $0 \in \mathrm{dom}(F^\alpha)$ (the purpose of the last assumption is to ensure that different F^α's have some common element in their domains). We abbreviate the tree ordering on T^α as $<_\alpha$.

By a standard thinning out argument, we can fix sets Z, A, T, and functions $f_{\alpha,\beta}$ and $g_{\alpha,\beta}$ for all $\alpha < \beta$ in Z satisfying:

(1) Z is an uncountable subset of the set of $\xi < \omega_1$ such that $\omega \cdot \xi = \xi$;
(2) $A \subseteq \omega_2$ is finite;
(3) T is a standard finite tree;
(4) for all $\alpha < \beta$ in Z:
   (a) $T^\alpha \restriction \alpha = T^\beta \restriction \beta = T$ and $T^\alpha \subseteq \beta$;
   (b) $f_{\alpha,\beta}$ is an isomorphism from $(T^\alpha, <_\alpha)$ to $(T^\beta, <_\beta)$ which is the identity function on T;
   (c) $f_{\alpha,\beta}(x^\alpha) = x^\beta$;
   (d) $\mathrm{dom}(F^\alpha) \cap \mathrm{dom}(F^\beta) = A$;
   (e) $g_{\alpha,\beta} : \mathrm{dom}(F^\alpha) \to \mathrm{dom}(F^\beta)$ is a bijection which is the identity function on A;
   (f) for all $\tau \in \mathrm{dom}(F^\alpha)$, $m \in \{-1, 1\}$, and $x \in T^\alpha$,
$$x \in \mathrm{dom}(F^\alpha(\tau)^m) \iff f_{\alpha,\beta}(x) \in \mathrm{dom}(F^\beta(g_{\alpha,\beta}(\tau))^m),$$
   and in that case,
$$f_{\alpha,\beta}(F^\alpha(\tau)^m(x)) = F^\beta(g_{\alpha,\beta}(\tau))^m(f_{\alpha,\beta}(x)).$$

Note that by property 4f, for all $\tau \in \mathrm{dom}(F^\alpha)$, $F^\alpha(\tau) \restriction T = F^\beta(g_{\alpha,\beta}(\tau)) \restriction T$, and in particular, for all $\tau \in A$, $F^\alpha(\tau) \restriction T = F^\beta(\tau) \restriction T$. Let $f_{\beta,\alpha}$ and $g_{\beta,\alpha}$ denote the inverse functions of $f_{\alpha,\beta}$ and $g_{\alpha,\beta}$ respectively. These two inverse functions satisfy properties 4b, 4c, 4e, and 4f when α and β are switched.

By the special property of ρ, fix $\alpha < \beta$ in Z larger than $\max\{\rho(\nu, \xi) : \nu, \xi \in A\}$ such that for all $\zeta \in \mathrm{dom}(F^\alpha) \setminus \mathrm{dom}(F^\beta)$, for all $\tau \in \mathrm{dom}(F^\beta) \setminus \mathrm{dom}(F^\alpha)$, and for all $\gamma \in \mathrm{dom}(F^\alpha) \cap \mathrm{dom}(F^\beta) = A$,
$$\rho(\zeta, \tau) \geq \max\{\rho(\zeta, \gamma), \rho(\tau, \gamma), \max(\mathrm{ht}[T])\}.$$
Note that since $0 \in \mathrm{dom}(F^\alpha) \cap \mathrm{dom}(F^\beta)$, for all ζ and τ as above, $\rho(\zeta, \tau) \geq \max(\mathrm{ht}[T])$. The goal for the rest of the proof is to find a condition (W, H) which extends the conditions (T^α, F^α) and (T^β, F^β) such that $x^\alpha <_W x^\beta$.

Claim 1: $f_{\alpha,\beta}(x^\alpha \restriction \alpha) = x^\beta \restriction \beta$.

Proof. Since $f_{\alpha,\beta}$ is an isomorphism, $x^\alpha \restriction \alpha \leq_\alpha x^\alpha$ implies that $f_{\alpha,\beta}(x^\alpha \restriction \alpha) \leq_\beta f_{\alpha,\beta}(x^\alpha) = x^\beta$. Since $f_{\alpha,\beta}$ maps level α of T^α onto level β of T^β, it follows that $f_{\alpha,\beta}(x^\alpha \restriction \alpha) = x^\beta \restriction \beta$. □

Let X_α be the closure of the singleton $\{x^\alpha \restriction \alpha\}$ under the functions in $\{F^\alpha(\tau)^m : \tau \in A,\ m \in \{-1,1\}\}$, and let X_β be the closure of $\{x^\beta \restriction \beta\}$ under the functions in $\{F^\beta(\tau)^m : \tau \in A,\ m \in \{-1,1\}\}$.

Claim 2: There exists an injective sequence $\langle z_0, \dots, z_{n-1}\rangle$ which lists X_α satisfying that for any $l < n$, there exists a decreasing sequence $i_0 > \dots > i_q$ such that $i_0 = l$, $i_q = 0$, and for all $j < q$, there exists some $(m, \zeta) \in \{-1,1\} \times A$ such that $F^\alpha(\zeta)^m(z_{i_j}) = z_{i_{j+1}}$. Moreover, $X_\beta = \{f_{\alpha,\beta}(z_i) : i < n\}$, so $f_{\alpha,\beta} \restriction X_\alpha$ is a bijection between X_α and X_β.

Proof. We construct the sequence $\langle z_0, \dots, z_{n-1}\rangle$ by induction. Let $z_0 = x_\alpha \restriction \alpha$. Now assume that $\langle z_0, \dots, z_p\rangle$ is defined as required. Apply one at a time the functions in $\{F^\alpha(\tau)^m : \tau \in A,\ m \in \{-1,1\}\}$ to the members of $\{z_0, \dots, z_p\}$ until we obtain a new element not in $\{z_0, \dots, z_p\}$, which we denote by z_{p+1}. If this process does not result in any new element, then we are done and in that case $X_\alpha = \{z_0, \dots, z_p\}$. This completes the construction. The first property described of this sequence can be easily proven by induction, and the statement about X_β follows from Claim 1 and property 4f. □

Claim 3: If a and b are distinct elements of X_α, then there exists an injective sequence $\langle x_0, \dots, x_{k-1}\rangle$ consisting of elements of X_α such that $k \geq 2$, $x_0 = a$, $x_{k-1} = b$, and for all $i < k-1$ there exists some $(m, \tau) \in \{-1,1\} \times A$ such that $F^\alpha(\tau)^m(x_i) = x_{i+1}$.

Proof. Fix a sequence $\langle z_0, \dots, z_{n-1}\rangle$ as described in Claim 2. Fix $l_a, l_b < n$ such that $a = z_{l_a}$ and $b = z_{l_b}$. For each $c \in \{a, b\}$ fix a decreasing sequence $i^c_0 > \dots > i^c_{q_c}$ such that $i^c_0 = l_c$, $i^c_{q_c} = 0$, and for all $j < q_c$, there exists some $(m, \zeta) \in \{-1,1\} \times A$ such that $F^\alpha(\zeta)^m(z_{i^c_j}) = z_{i^c_{j+1}}$.

Now consider the concatenation of the sequence $\langle z_{i^a_0}, \dots, z_{i^a_{q_a}}\rangle$ with the reverse of the sequence $\langle z_{i^b_0}, \dots, z_{i^b_{q_b-1}}\rangle$. This sequence starts at a, ends at b, and each of its elements has some relation to its adjacent elements with respect to $\{F^\alpha(\tau) : \tau \in A\}$. So we are done provided that this concatenated sequence is injective. If it is not, then we adjust it by deleting repetitions one at a time going from left to right: each time we encounter a subsequence of our current sequence of the form $\langle d, \dots, d\rangle$, chosen as big as possible so that the last member of this subsequence is the last occurence of d in the current sequence, remove the elements of this subsequence after its first element. Continue in this manner moving from left to right until all repetitions are deleted and we obtain an injective sequence. Note that after every step of this process, the adjusted sequence still starts with a and ends with b, and adjacent elements are still related as required. □

Claim 4: If a and b are distinct elements of X_α and there exists some $\tau \in \mathrm{dom}(F^\alpha) \setminus A$ and $m \in \{-1,1\}$ such that $F^\alpha(\tau)^m(a) = b$, then there exists some $\sigma \in A$ and $l \in \{-1,1\}$ such that $F^\alpha(\sigma)^l(a) = b$.

Proof. Apply Claim 3 to fix an injective sequence $\langle x_0, \dots, x_{k-1}\rangle$ consisting of elements of X_α such that $k \geq 2$, $x_0 = a$, $x_{k-1} = b$, and for all $i < k-1$ there exists some $(m, \tau) \in \{-1,1\} \times A$ such that $F^\alpha(\tau)^m(x_i) = x_{i+1}$. Assume that for some $\tau \in \mathrm{dom}(F^\alpha) \setminus A$ and $m \in \{-1,1\}$, $F^\alpha(\tau)^m(a) = b$. If $k > 2$, then $\langle x_0, \dots, x_{k-1}, x_0\rangle$ is a loop, contradicting that $F^\alpha(\tau)$ is ρ-separated on T^α_α. Hence, $k = 2$, so $a = x_0$ and $b = x_1$, which easily implies the conclusion of the claim. □

To prepare for the amalgamation of the conditions (T^α, F^α) and (T^β, F^β), we make two preliminary steps. First, we extend the tree T^α to a tree $(T^\alpha)^+$, and secondly, we extend the condition $((T^\alpha)^+, F^\alpha)$ to a condition (U, G). For the first step, for each $y \in T^\beta_\beta \setminus X_\beta$ we add to T^α a chain C_y above $y \restriction \max(\mathrm{ht}[T])$, disjoint from T^α, consisting of elements of every possible height in $\mathrm{ht}[T^\alpha] \setminus \alpha$. Moreover, we arrange that any two such chains are disjoint. Let $(T^\alpha)^+$ be the tree thus formed. It is routine to check that $(T^\alpha)^+$ is a standard finite tree which is normal and satisfies that $\mathrm{ht}[(T^\alpha)^+] = \mathrm{ht}[T^\alpha]$. By Lemma 5.2, $((T^\alpha)^+, F^\alpha)$ is a condition extending (T^α, F^α). Note that $(T^\alpha)^+ \restriction \alpha = T$ and for all $\tau \in \mathrm{dom}(F^\alpha)$, the domain and range of $F^\alpha(\tau)$ is disjoint from $(T^\alpha)^+ \setminus T^\alpha$.

Applying the fact that T^α is normal, fix an element z^α on the top level of T^α such that $x^\alpha \leq_\alpha z^\alpha$. Since $\{\rho(\nu, \xi) : \nu, \xi \in A\} \subseteq \alpha$, it follows by Lemma 4.9 that $\{F^\alpha(\tau) : \tau \in A\}$ is separated on X_α. Applying Corollary 6.4, fix $(U, G) \in \mathbb{P}$ and $X^+_\alpha \subseteq U_{\max(\mathrm{ht}[T^\alpha])}$ satisfying:

- $(U, G) \leq ((T^\alpha)^+, F^\alpha)$;
- $\mathrm{ht}[T^\alpha] = \mathrm{ht}[U]$ and $\mathrm{dom}(F^\alpha) = \mathrm{dom}(G)$;
- $U \setminus (T^\alpha)^+ \subseteq \bigcup\{\mathrm{Succ}_U(x) : x \in X_\alpha\}$;
- for all $\tau \in \mathrm{dom}(G)$ and for all $z \in \mathrm{dom}(G(\tau)) \setminus \mathrm{dom}(F^\alpha(\tau))$, both z and $G(\tau)(z)$ are in $\bigcup\{\mathrm{Succ}_U(x) : x \in X_\alpha\}$;
- X^+_α has unique drop-downs to α, $X^+_\alpha \restriction \alpha = X_\alpha$, and $z^\alpha \in X^+_\alpha$;
- for all $\tau \in A$, X_α and X^+_α are $G(\tau)$-consistent.

Note that $U \restriction \alpha = T$ and for all $\tau \in \mathrm{dom}(G)$, $G(\tau) \restriction T = F^\alpha(\tau) \restriction T$.

To deal with the complexities of what follows, we split the elements of $U \setminus \alpha$ into three disjoint sets. Let $\mathcal{C}$ denote the set of elements belonging to some chain C_y, where $y \in T^\beta_\beta \setminus X_\beta$; in other words, $\mathcal{C} = (T^\alpha)^+ \setminus T^\alpha$. Observe that for all $\tau \in \mathrm{dom}(G)$, the domain and range of $G(\tau)$ is disjoint from $\mathcal{C}$. Let $\mathcal{S}$ denote the set of $z \in U \setminus \alpha$ such that for some $y \in X^+_\alpha$, $z = y \restriction \delta$ for some $\delta \in \mathrm{ht}[U] \setminus \alpha$. In other words, $\mathcal{S} = \bigcup\{X^+_\alpha \restriction \delta : \delta \in \mathrm{ht}[U] \setminus \alpha\}$. Finally, let $\mathcal{D}$ denote the set of elements in $U \setminus \alpha$ which are not in $\mathcal{C}$ or $\mathcal{S}$.

Recall that X^+_α has unique drop-downs to α, $X^+_\alpha \restriction \alpha = X_\alpha$, and for all $\tau \in A$, X_α and X^+_α are $G(\tau)$-consistent. It follows by Lemma 4.2 that for all $\delta \in \mathrm{ht}[U] \setminus \alpha$, $\mathcal{S} \cap U_\delta = X^+_\alpha \restriction \delta$ has unique drop-downs to α, $(\mathcal{S} \cap U_\delta) \restriction \alpha = X_\alpha$, and for all $\tau \in A$, X_α and $\mathcal{S} \cap U_\delta$ are $G(\tau)$-consistent.

Claim 5: If a and b are distinct elements of $\mathcal{S}$ of the same height, then there exists an injective sequence $\langle y_0, \ldots, y_{k-1}\rangle$ consisting of elements of $\mathcal{S}$ such that $k \geq 2$, $y_0 = a$, $y_{k-1} = b$, and for all $i < k - 1$ there exists some $(m, \tau) \in \{-1, 1\} \times A$ such that $G(\tau)^m(y_i) = y_{i+1}$.

Proof. Let δ be the height of a and b. Apply Claim 3 to $a \restriction \alpha$ and $b \restriction \alpha$ to obtain an injective sequence $\langle x_0, \ldots, x_{k-1}\rangle$ consisting of elements of X_α such that $k \geq 2$, $x_0 = a \restriction \alpha$, $y_{k-1} = b \restriction \alpha$, and for all $i < k - 1$ there exists some $(m, \tau) \in \{-1, 1\} \times A$ such that $F^\alpha(\tau)^m(x_i) = x_{i+1}$, that is, $G(\tau)^m(x_i) = x_{i+1}$. For each $0 < i < k$, let y_i be the unique element of $\mathcal{S} \cap U_\delta$ above x_i. Note that $y_0 = a$ and $y_{k-1} = b$. For all $\tau \in A$, X_α and $\mathcal{S} \cap U_\delta$ are $G(\tau)$-consistent, so the desired conclusion clearly holds. □

Now we begin the definition of the condition (W, H). This is done in two steps: first we construct W, and then we construct H.

Step 1: Constructing W.

We amalgamate the trees U and T^β into a tree W with underlying set $U \cup T^\beta$ as follows. Since $U \restriction \alpha = T^\beta \restriction \beta = T$, it suffices to specify for each $y \in T^\beta_\beta$ an immediate predecessor y^- of y in $U_{\max(\mathrm{ht}[U])}$ such that $y \restriction_{T^\beta} \max(\mathrm{ht}[T]) = y^- \restriction_U \max(\mathrm{ht}[T])$. Moreover, we can ensure that W a simple extension of T^β by arranging that the function $y \mapsto y^- \restriction \alpha$ is injective. For each $y \in T^\beta_\beta \setminus X_\beta$, let y^- be the top element of the chain C_y. For each $z \in X_\beta$, let z^- be the unique element of X^+_α which is above $f_{\beta,\alpha}(z)$ in U. It is easy to check that this works.

Note that by definition, for all $y \in X_\beta$, $y \restriction_W \alpha = f_{\beta,\alpha}(y)$. Now z^α is the unique element of X^+_α which is greater than or equal to $x^\alpha \restriction \alpha$ in U. By Claim 1, $x^\alpha \restriction \alpha = f_{\beta,\alpha}(x^\beta \restriction \beta)$. So $(x^\beta \restriction \beta)^- = z^\alpha$. Therefore, $x^\alpha <_W x^\beta$ as desired.

Note that the elements of $\mathcal{D}$ do not have anything above them in $W \setminus \beta$. For each $z \in \mathcal{C}$, z is in the chain C_y for some $y \in T^\beta_\beta \setminus X_\beta$, and $z <_W y$ by definition. For each $z \in \mathcal{S}$, $z <_W f_{\alpha,\beta}(z \restriction \alpha)$. For every $z \in \mathcal{C} \cup \mathcal{S}$, let z^+ be the unique member of W_β above z. Note that $z \in \mathcal{C}$ iff $z^+ \in W_\beta \setminus X_\beta$, and $z \in \mathcal{S}$ iff $z^+ \in X_\beta$. Also, if $z \in \mathcal{S}$ then $z \restriction \alpha = f_{\beta,\alpha}(z^+)$, that is, $f_{\alpha,\beta}(z \restriction \alpha) = z^+$.

For each $\tau \in \mathrm{dom}(F^\beta)$, let $\bar{F}^\beta(\tau)$ be the downward closure of $F^\beta(\tau)$ in W. Since W is a simple extension of T^β, it follows by Lemma 3.8 that $\bar{F}^\beta(\tau)$ is a standard function on W and $\bar{F}^\beta(\tau) \restriction T^\beta = F^\beta(\tau)$. In particular, $\bar{F}^\beta(\tau) \restriction T = F^\beta(\tau) \restriction T$. Observe that the domain and range of $\bar{F}^\beta(\tau)$ is disjoint from $\mathcal{D}$. Also,

$$\forall x, y \in \mathcal{C} \cup \mathcal{S}\ (\bar{F}^\beta(\tau)(x) = y \iff F^\beta(\tau)(x^+) = y^+).$$

Claim 6: For all $\tau \in A$, $\bar{F}^\beta(\tau) \restriction T = G(\tau) \restriction T$.

Proof. We have that $\bar{F}^\beta(\tau) \restriction T = F^\beta(\tau) \restriction T = F^\alpha(\tau) \restriction T = G(\tau) \restriction T$ (where property 4f is used for the second equality). □

Claim 7: For all $(m, \tau) \in \{-1, 1\} \times A$, $\mathcal{C}$ and $\mathcal{S}$ are both closed under $\bar{F}^\beta(\tau)^m$.

Proof. Assume that $\bar{F}^\beta(\tau)^m(x) = y$, where $x \in \mathcal{C} \cup \mathcal{S}$. Since the domain and range of $\bar{F}^\beta(\tau)$ are disjoint from $\mathcal{D}$, $y \in \mathcal{C} \cup \mathcal{S}$. So $F^\beta(\tau)^m(x^+) = y^+$ by the above. If $x \in \mathcal{S}$, then $x^+ \in X_\beta$. Since X_β is closed under $F^\beta(\tau)^m$, $y^+ \in X_\beta$, and hence $y \in \mathcal{S}$. If $x \in \mathcal{C}$, then $x^+ \in W_\beta \setminus X_\beta$. But X_β is closed under $F^\beta(\tau)^{-m}$, so $y^+ \in W_\beta \setminus X_\beta$. Thus, $y \in \mathcal{C}$. □

Claim 8: For all $(m, \tau) \in \{-1, 1\} \times A$, $\bar{F}^\beta(\tau)^m \restriction \mathcal{S} = G(\tau)^m \restriction \mathcal{S}$.

Proof. Suppose that $x \in \mathcal{S}$ and $\bar{F}^\beta(\tau)^m(x) = y$. Then $y \in \mathcal{S}$ by Claim 7, $y^+ \in X_\beta$, and $F^\beta(\tau)^m(x^+) = y^+$. We also have that $y \restriction \alpha = f_{\beta,\alpha}(y^+) = f_{\beta,\alpha}(F^\beta(\tau)^m(x^+)) = F^\alpha(\tau)^m(f_{\beta,\alpha}(x^+)) = F^\alpha(\tau)^m(x \restriction \alpha) = G(\tau)^m(x \restriction \alpha)$. By $G(\tau)$-consistency, it follows that $G(\tau)^m(x) = y$.

On the other hand, assume that $x \in \mathcal{S}$ and $G(\tau)^m(x) = y$. Then $F^\alpha(\tau)^m(x \restriction \alpha) = y \restriction \alpha$. Since X_α is closed under $F^\alpha(\tau)^m$, $y \restriction \alpha \in X_\alpha$. Let y' be the unique element of $\mathcal{S}$ above $y \restriction \alpha$ with the same height as x. Then by $G(\tau)$-consistency, $G(\tau)^m(x) = y'$. Hence,

$y = y'$ and so $y \in \mathcal{S}$. By property 4f, $y^+ = f_{\alpha,\beta}(y \restriction \alpha) = f_{\alpha,\beta}(F^\alpha(\tau)^m(x \restriction \alpha)) = F^\beta(\tau)^m(f_{\alpha,\beta}(x \restriction \alpha)) = F^\beta(\tau)^m(x^+)$. Hence, $y = \bar{F}^\beta(\tau)^m(x)$. □

The following two claims record for future reference observations which we already made above.

Claim 9: For all $\tau \in \mathrm{dom}(G)$, the domain and range of $G(\tau)$ are disjoint from $\mathcal{C}$.

Claim 10: For all $\tau \in \mathrm{dom}(F^\beta)$, the domain and range of $\bar{F}^\beta(\tau)$ are disjoint from $\mathcal{D}$.

By Lemma 5.2, $(W, G) \in \mathbb{P}$ and $(W, G) \leq (U, G)$. By Lemma 5.3, $(W, \bar{F}^\beta) \in \mathbb{P}$ and $(W, \bar{F}^\beta) \leq (T^\beta, F^\beta)$. So it suffices to construct H so that $(W, H) \in \mathbb{P}$ and (W, H) extends both (W, G) and $(W, \bar{F}^\beta)$.

Step 2: Constructing H.

Let the domain of H be equal to $\mathrm{dom}(G) \cup \mathrm{dom}(F^\beta)$. If $\tau \in \mathrm{dom}(G) \setminus \mathrm{dom}(F^\beta)$, then let $H(\tau) = G(\tau)$. If $\tau \in \mathrm{dom}(F^\beta) \setminus \mathrm{dom}(G)$, then let $H(\tau) = \bar{F}^\beta(\tau)$. Now suppose that $\tau \in \mathrm{dom}(G) \cap \mathrm{dom}(F^\beta)$. Then $\tau \in \mathrm{dom}(F^\alpha) \cap \mathrm{dom}(F^\beta) = A$. We claim that for all $x \in \mathrm{dom}(\bar{F}^\beta(\tau)) \cap \mathrm{dom}(G(\tau))$, $\bar{F}^\beta(\tau)(x) = G(\tau)(x)$. It then easily follows that $H(\tau) = \bar{F}^\beta(\tau) \cup G(\tau)$ is a strictly increasing, level preserving, downwards closed partial function from W to W with no fixed-points other than 0. So let $x \in \mathrm{dom}(\bar{F}^\beta(\tau)) \cap \mathrm{dom}(G(\tau))$. By Claim 6, we can assume that $x \notin T$. Also, $x \in \mathrm{dom}(G(\tau))$ implies that $x \in U$. By Claims 9 and 10, $x \in \mathcal{S}$. By Claim 8, $\bar{F}^\beta(\tau)(x) = G(\tau)(x)$.

Now we show that $H(\tau)$ is injective. Since $\bar{F}^\beta(\tau)$ and $G(\tau)$ are each injective, it is enough to show that for all $x \in \mathrm{dom}(\bar{F}^\beta(\tau)) \setminus \mathrm{dom}(G(\tau))$ and for all $y \in \mathrm{dom}(G(\tau)) \setminus \mathrm{dom}(\bar{F}^\beta(\tau))$, $\bar{F}^\beta(\tau)(x) \neq G(\tau)(y)$. As $\bar{F}^\beta(\tau) \restriction T = G(\tau) \restriction T$ by Claim 6 and $H(\tau)$ is level preserving, it suffices to consider x and y which are both in U_δ for some $\delta \in \mathrm{ht}[U] \setminus \alpha$. Suppose for a contradiction that $\bar{F}^\beta(\tau)(x) = z = G(\tau)(y)$. By Claims 9 and 10, $z \in \mathcal{S}$. By Claim 7, x is in $\mathcal{S}$. By Claim 8, x is in the domain of $G(\tau)$, which is a contradiction. This completes the proof that $H(\tau)$ is injective and hence is a standard function on W.

We need two more claims.

Claim 11: If $a, b \in \mathcal{S}$, $\tau \in \mathrm{dom}(H)$, and $H(\tau)(a) = b$, then there exists some $(m, \zeta) \in \{-1, 1\} \times \mathrm{dom}(F^\beta)$ such that $F^\beta(\zeta)^m(a^+) = b^+$.

Proof. The equation $H(\tau)(a) = b$ means that either $\bar{F}^\beta(\tau)(a) = b$ or $G(\tau)(a) = b$. In the former case, $F^\beta(\tau)(a^+) = b^+$ and we are done. Suppose that $G(\tau)(a) = b$. By Claim 8, we may assume that $\tau \notin A$, otherwise we are in the case just considered. Since $G(\tau)(a) = b$, it follows that $G(\tau)(a \restriction \alpha) = b \restriction \alpha$, that is, $F^\alpha(\tau)(a \restriction \alpha) = b \restriction \alpha$. By Claim 4, there exists some $(m, \xi) \in \{-1, 1\} \times A$ such that $F^\alpha(\xi)^m(a \restriction \alpha) = b \restriction \alpha$. By property 4f, $F^\beta(\xi)^m(a^+) = F^\beta(\xi)^m(f_{\alpha,\beta}(a \restriction \alpha)) = f_{\alpha,\beta}(F^\alpha(\xi)^m(a \restriction \alpha)) = f_{\alpha,\beta}(b \restriction \alpha) = b^+$. □

Claim 12: If $a, b \in \mathcal{S}$, $\gamma \in \mathrm{dom}(F^\beta)$, and $\bar{F}^\beta(\gamma)(a) = b$, then there exists some $(m, \xi) \in \{-1, 1\} \times \mathrm{dom}(G)$ such that $G(\xi)^m(a) = b$.

Proof. By Claim 8, we are done if $\gamma \in A$. So assume that $\gamma \in \mathrm{dom}(F^\beta) \setminus \mathrm{dom}(G)$. Then neither γ nor $g_{\beta,\alpha}(\gamma)$ are in A. We have that $F^\beta(\gamma)(a^+) = b^+$, so $F^\alpha(g_{\beta,\alpha}(\gamma))(a \restriction \alpha) = F^\alpha(g_{\beta,\alpha}(\gamma))(f_{\beta,\alpha}(a^+)) = f_{\beta,\alpha}(F^\beta(\gamma)(a^+)) = f_{\beta,\alpha}(b^+) = b \restriction \alpha$. By Claim 4, there exists some $(m, \xi) \in \{-1, 1\} \times A$ such that $G(\xi)^m(a \restriction \alpha) = b \restriction \alpha$. By $G(\xi)$-consistency, $G(\xi)^m(a) = b$. □

We have now constructed (W, H). To complete the proof, we need to show that $(W, H) \in \mathbb{P}$ and (W, H) extends both (W, G) and $(W, \bar{F}^\beta)$. For proving the former, it suffices to show that for all $\delta \in \mathrm{ht}[W]$ greater than 0, H is ρ-separated on W_δ. Once we know that $(W, H) \in \mathbb{P}$, in order to prove that (W, H) extends (W, G) and $(W, \bar{F}^\beta)$, it suffices to verify Definition 5.1(c) in each case.

Fix $\delta \in \mathrm{ht}[W]$ greater than 0 and we prove that H is ρ-separated on W_δ using Proposition 4.14 (Characterization of ρ-Separation). Note that if $\delta \geq \beta$, then since $W \setminus \beta = T^\beta \setminus \beta$ and H and F^β are the same on $W \setminus \beta$, we are done because (T^β, F^β) is a condition. So assume that $\delta < \beta$.

Verifying property (1) of Proposition 4.14 (Characterization of ρ-Separation):

Suppose that $x, y \in W_\delta$ and (m_0, τ_0) and (m_1, τ_1) are distinct pairs in $\{-1, 1\} \times \mathrm{dom}(H)$ satisfying that $H(\tau_0)^{m_0}(x) = y$ and $H(\tau_1)^{m_1}(x) = y$. We prove that $\rho(\tau_0, \tau_1) \geq \delta$. Since (W, G) and $(W, \bar{F}^\beta)$ are conditions, without loss of generality we may assume that $G(\tau_0)^{m_0}(x) = y$ and $\bar{F}^\beta(\tau_1)^{m_1}(x) = y$.

First, suppose that $\delta < \alpha$. Then $W_\delta = T^\alpha_\delta$. Recall that G is the same as F^α on T and $\bar{F}^\beta$ is the same as F^β on T. So $F^\alpha(\tau_0)^{m_0}(x) = y$ and $F^\beta(\tau_1)^{m_1}(x) = y$. If for some $k < 2$, $\tau_k \in A$, then $F^\alpha(\tau_k) \restriction T = F^\beta(\tau_k) \restriction T$ so we are done since (T^α, F^α) and (T^β, F^β) are conditions. So assume that $\tau_0 \in \mathrm{dom}(F^\alpha) \setminus \mathrm{dom}(F^\beta)$ and $\tau_1 \in \mathrm{dom}(F^\beta) \setminus \mathrm{dom}(F^\alpha)$. Then by the choice of α and β, $\rho(\tau_0, \tau_1) \geq \max(\mathrm{ht}[T]) \geq \delta$.

Secondly, assume that $\alpha \leq \delta < \beta$. By Claims 9 and 10, x and y are in $\mathcal{S}$. By Claim 8, we may assume that neither τ_0 nor τ_1 are in A, for otherwise we are done since (W, G) and $(W, \bar{F}^\beta)$ are conditions. So $\tau_0 \in \mathrm{dom}(G) \setminus \mathrm{dom}(F^\beta)$ and $\tau_1 \in \mathrm{dom}(F^\beta) \setminus \mathrm{dom}(G)$. In particular, $\tau_0 \notin A$. As $F^\alpha(\tau_0)^{m_0}(x \restriction \alpha) = G(\tau_0)^{m_0}(x \restriction \alpha) = y \restriction \alpha$, by Claim 4 there exists some $(n, \sigma) \in \{-1, 1\} \times A$ such that $G(\sigma)^n(x \restriction \alpha) = y \restriction \alpha$. By $G(\sigma)$-consistency, $G(\sigma)^n(x) = y$. Hence, $\rho(\tau_0, \sigma) \geq \delta$. Therefore, $\rho(\tau_0, \tau_1) \geq \rho(\tau_0, \sigma) \geq \delta$.

Verifying property (2) of Proposition 4.14 (Characterization of ρ-Separation):

Suppose for a contradiction that there exists a loop $\langle a_0, \ldots, a_{n-1} \rangle$ of elements of W_δ with respect to H. So $n \geq 4$, the sequence $\langle a_0, \ldots, a_{n-2} \rangle$ is injective, $a_0 = a_{n-1}$, and for all $i < n-1$ there exists $(m_i, \tau_i) \in \{-1, 1\} \times \mathrm{dom}(H)$ such that $H(\tau_i)^{m_i}(a_i) = a_{i+1}$. For each $i < n$, either $G(\tau_i)^{m_i}(a_i) = a_{i+1}$ or $\bar{F}^\beta(\tau_i)^{m_i}(a_i) = a_{i+1}$.

First, assume that $\delta < \alpha$. Since G is equal to F^α on T and $\bar{F}^\beta$ is equal to F^β on T, for each $i < n$, either $F^\alpha(\tau_i)^{m_i}(a_i) = a_{i+1}$ or $F^\beta(\tau_i)^{m_i}(a_i) = a_{i+1}$. Let $i < n$. If $\tau_i \in A$, then $F^\alpha(\tau_i) \restriction T = F^\beta(\tau_i) \restriction T$, so in either case, $F^\alpha(\tau_i)^{m_i}(a_i) = a_{i+1}$. Suppose that it is not the case that $F^\alpha(\tau_i)^{m_i}(a_i) = a_{i+1}$. Then $F^\beta(\tau_i)^{m_i}(a_i) = a_{i+1}$. By property 4f, $F^\alpha(g_{\beta,\alpha}(\tau_i))^{m_i}(a_i) = a_{i+1}$. It follows that $\langle a_0, \ldots, a_{n-1} \rangle$ is a loop in T^α_δ with respect to F^α, which contradicts that (T^α, F^α) is a condition.

Secondly, assume that $\alpha \leq \delta < \beta$. We consider four cases.

Case 1: For all $i < n$, $a_i \in \mathcal{C} \cup \mathcal{S}$. Consider $i < n-1$. Then either $\bar{F}^\beta(\tau_i)^{m_i}(a_i) = a_{i+1}$ or $G(\tau_i)^{m_i}(a_i) = a_{i+1}$. If either a_i or a_{i+1} is in $\mathcal{C}$, then we are in the former case by Claim 9. Hence, $F^\beta(\tau_i)^{m_i}(a_i^+) = a_{i+1}^+$. Otherwise, both a_i and a_{i+1} are in $\mathcal{S}$. By Claim 11, there exists some $(m, \zeta) \in \{-1, 1\} \times \mathrm{dom}(F^\beta)$ such that $F^\beta(\zeta)^m(a_i^+) = a_{i+1}^+$. So $\langle a_0^+, \ldots, a_{n-1}^+ \rangle$ is a loop in T^β_β with respect to F^β, which contradicts that (T^β, F^β) is a condition.

Case 2: For all $i < n$, $a_i \in \mathcal{D} \cup \mathcal{S}$. Let $i < n-1$. If either a_i or a_{i+1} is in $\mathcal{D}$, then $G(\tau_i)^{m_i}(a_i) = a_{i+1}$ by Claim 10. Otherwise, a_i and a_{i+1} are both in $\mathcal{S}$, so by Claim 12 there exists some $(m, \xi) \in \{-1, 1\} \times \mathrm{dom}(G)$ such that $G(\xi)^m(a_i) = a_{i+1}$. It follows that $\langle a_0, \ldots, a_{n-1} \rangle$ is a loop in U_δ with respect to G, which contradicts that (U, G) is a condition.

Case 3: For all $i < n$, $a_i \in \mathcal{C} \cup \mathcal{D}$. By Claims 9 and 10, there are no relations between members of $\mathcal{C}$ and members of $\mathcal{D}$ with respect to H. So either for all $i < n$, $a_i \in \mathcal{C}$, or for all $i < n$, $a_i \in \mathcal{D}$. So we are in either Case 1 or Case 2, which were already handled.

Case 4: The sequence $\langle a_0, \ldots, a_{n-1} \rangle$ contains at least one member in each of $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{S}$. Since there are no relations between members of $\mathcal{C}$ and members of $\mathcal{D}$ with respect to H, there do not exist adjacent elements of the loop where one is in $\mathcal{C}$ and the other is in $\mathcal{D}$. By shifting the sequence if necessary, we may assume without loss of generality that $a_0 \in \mathcal{C}$ and $a_1 \in \mathcal{S}$. Since $n \geq 4$, we know that a_0, a_1, and a_{n-2} are all distinct. Let $k < n-1$ be largest such that a_k is not in $\mathcal{C}$. So every member of the sequence after a_k is in $\mathcal{C}$, and in particular, $a_{k+1} \in \mathcal{C}$, so $a_k \in \mathcal{S}$. Moreover, a_1 and a_k are different because the loop contains at least one element which is in $\mathcal{D}$, and that member of $\mathcal{D}$ must be between a_1 and a_k. Apply Claim 5 to fix an injective sequence $\langle b_0, \ldots, b_{p-1} \rangle$ of elements of $\mathcal{S}$ such that $p \geq 2$, $b_0 = a_1$, $b_{p-1} = a_k$, and for all $j < p-1$ there exists $(m, \sigma) \in \{-1, 1\} \times A$ such that $G(\sigma)^m(b_i) = b_{i+1}$.

Now consider the sequence

$$\langle a_0, a_1, b_1, \ldots, b_{p-2}, a_k, a_{k+1}, \ldots, a_{n-1} \rangle.$$

Note that this sequence has length at least 4, all of the elements of this sequence are in $\mathcal{C} \cup \mathcal{S}$, any two adjacent elements of this sequence are related with respect to H, and this sequence minus the last element is injective. Thus, we have a loop in W_δ with respect to H consisting of members of $\mathcal{C} \cup \mathcal{S}$. So we are in Case 1, which was already handled.

This completes the proof that (W, H) is a condition.

It remains to prove that (W, H) extends (W, G) and $(W, \bar{F}^\beta)$. In both cases, it suffices to verify Definition 5.1(c).

Proving that $(W, H) \leq (W, \bar{F}^\beta)$*:*

Suppose that γ and τ are distinct elements of $\mathrm{dom}(F^\beta)$, $x \in \mathrm{dom}(H(\gamma)) \cap \mathrm{dom}(H(\tau))$, and $H(\gamma)(x) = H(\tau)(x)$. We prove that there exists some $z \in W$ such that $x \leq_W z$ and $\bar{F}^\beta(\gamma)(z) = \bar{F}^\beta(\tau)(z)$. If $x \in W \setminus \beta$, then $H(\gamma) = F^\beta(\gamma)$ and $H(\tau) = F^\beta(\tau)$, so we are done. Assume that $x \in T$. For each $\xi \in \{\gamma, \tau\}$, $H(\xi)(x)$ is equal to either $F^\alpha(\xi)(x)$ or $F^\beta(\xi)(x)$. Moreover, by property 4f we are in both cases when $\xi \in A$. However, since $\xi \in \mathrm{dom}(F^\beta)$, if $H(\xi)(x) = F^\alpha(\xi)(x)$, then $\xi \in A$. So no matter what, $H(\gamma)(x) = F^\beta(\gamma)(x)$ and $H(\tau)(x) = F^\beta(\tau)(x)$, and we are done.

Now assume that $x \in U_\delta$ for some $\alpha \leq \delta < \beta$. Let $y = H(\gamma)(x)$. So also $y = H(\tau)(x)$. Suppose that $x \in \mathcal{C}$. Then by Claim 9, $H(\gamma)(x) = \bar{F}^\beta(\gamma)(x)$ and $H(\tau)(x) = \bar{F}^\beta(\gamma)(x)$. Therefore, $F^\beta(\gamma)(x^+) = y^+ = F^\beta(\tau)(x^+)$ and we are done. If $x \in \mathcal{D}$, then by Claim 10, $H(\gamma)(x) = G(\gamma)(x)$ and $H(\tau)(x) = G(\tau)(x)$. So γ and τ are in $\mathrm{dom}(G) = \mathrm{dom}(F^\alpha)$, and hence are in A. So $\rho(\gamma, \tau) < \alpha$. Since G is ρ-separated on W_δ, the equations $G(\gamma)(x) = y$ and $G(\tau)(x) = y$ imply that $\rho(\gamma, \tau) \geq \delta \geq \alpha$, which is a contradiction.

Finally, assume that $x \in \mathcal{S}$. If $H(\gamma)(x) = \bar{F}^\beta(\gamma)(x)$ and $H(\tau)(x) = \bar{F}^\beta(\tau)(x)$, then $F^\beta(\gamma)(x^+) = y^+$ and $F^\beta(\tau)(x^+) = y^+$ and we are done. Otherwise, without loss of generality, $H(\gamma)(x) = G(\gamma)(x)$ and it is not the case that $H(\gamma)(x) = \bar{F}^\beta(\gamma)(x)$. But $H(\gamma)(x) = G(\gamma)(x)$ implies that $\gamma \in A$, and then Claim 8 implies $H(\gamma)(x) = \bar{F}^\beta(\gamma)(x)$, which is a contradiction. This completes the proof that $(W, H) \leq (W, \bar{F}^\beta)$.

Proving that $(W, H) \leq (W, G)$*:*

Suppose that γ and τ are distinct elements of $\mathrm{dom}(G)$, $x \in \mathrm{dom}(H(\gamma)) \cap \mathrm{dom}(H(\tau))$, and $H(\gamma)(x) = H(\tau)(x)$. Let $y = H(\gamma)(x)$. We prove that there exists some $z \in W$ such that $x \leq_W z$ and $G(\gamma)(z) = G(\tau)(z)$. If $H(\gamma)(x) = G(\gamma)(x)$ and $H(\tau)(x) = G(\tau)(x)$, then we are done. So assume without loss of generality that $H(\gamma)(x) = \bar{F}^\beta(\gamma)(x)$ and it is not the case that $H(\gamma)(x) = G(\gamma)(x)$. It follows that $\gamma \in A$. By Claim 6, we may assume that $x \notin T$. Assume that $x \in W_\delta = T^\beta_\delta$ for some $\beta \leq \delta$. Then $H(\tau)(x) = F^\beta(\tau)(x)$ and also $\tau \in A$. Since F^β is ρ-separated on T^β_δ, $\rho(\gamma, \tau) \geq \delta \geq \beta$. But γ and τ are both in A, so $\rho(\gamma, \tau) < \alpha < \delta$, which is a contradiction.

Finally, assume that $x \in W_\delta = U_\delta$ for some $\alpha \leq \delta < \beta$. If $x \in \mathcal{C}$, then by Claim 9, $H(\tau)(x) = \bar{F}^\beta(\tau)(x)$. So γ and τ are both in A, and hence $\rho(\gamma, \tau) < \alpha$. But $\bar{F}^\beta(\gamma)(x) = y = \bar{F}^\beta(\tau)(x)$ implies that $F^\beta(\gamma)(x^+) = y^+ = F^\beta(\tau)(x^+)$. Since F^β is ρ-separated on T^β_β, $\rho(\gamma, \tau) \geq \beta$, which is a contradiction. If $x \in \mathcal{D}$, then by Claim 10, $H(\gamma)(x) = G(\gamma)(x)$, which contradicts our assumption. Finally, assume that $x \in \mathcal{S}$. By Claim 8, $H(\gamma)(x) = \bar{F}^\beta(\gamma)(x) = G(\gamma)(x)$, which again contradicts our assumption. □

Corollary 7.2. *The forcing poset* $\mathbb{P}$ *is c.c.c.*

Proof. Let $\{(T^\alpha, F^\alpha) : \alpha < \omega_1\rangle$ be a family of conditions. Applying Lemma 5.4, without loss of generality we may assume that for all $\alpha < \omega_1$, $\alpha \in \mathrm{ht}[T^\alpha]$. For each $\alpha < \omega_1$, fix some $x^\alpha \in T^\alpha_\alpha$. Now apply Theorem 7.1 to find $\alpha < \beta < \omega_1$ such that (T^α, F^α) and (T^β, F^β) are compatible. □

Corollary 7.3. *The forcing poset* $\mathbb{P}$ *forces that* $T^{\dot{G}}$ *is Suslin.*

Proof. Suppose for a contradiction that some condition $p \in \mathbb{P}$ forces that there exists an uncountable antichain of $T^{\dot{G}}$. Then we can find a sequence of $\mathbb{P}$-names $\langle \dot{x}^\alpha : \alpha < \omega_1\rangle$ for elements of $T^{\dot{G}}$ such that p forces that for each $\alpha < \omega_1$, $\mathrm{ht}(\dot{x}^\alpha) \geq \alpha$, and for all $\alpha < \beta < \omega_1$, $\dot{x}^\alpha$ and $\dot{x}^\beta$ are incomparable. For each $\alpha < \omega_1$, pick a condition $(T^\alpha, F^\alpha) \leq p$ and some $x^\alpha \in T^\alpha$ such that (T^α, F^α) forces that $\dot{x}^\alpha$ is equal to $\check{x}^\alpha$. By Theorem 7.1, there are $\alpha < \beta < \omega_1$ and a condition (W, H) extending (T^α, F^α) and (T^β, F^β) such that $x^\alpha <_W x^\beta$. But this contradicts that (W, H) forces that $\dot{x}^\alpha$ and $\dot{x}^\beta$ are incomparable. □

We have now completed the proof of the main theorem.

We close the article with a question.

Question. *Is it consistent that there exists a strongly non-saturated Aronszajn tree and there does not exist a weak Kurepa tree?*

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