Commensurability of spherical Artin groups

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Artin groups

- S finite set of generators
- $M = (m_{s,t})_{s,t \in S}$ a symmetric matrix of $\{0, 1, \ldots, \infty\}$

The **Artin group** corresponding to (S, M) is given by the presentation:

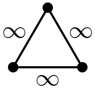
$$A = \left\langle\, S \;\middle|\; \underbrace{stst\ldots}_{m_{s,t}} = \underbrace{tsts\ldots}_{m_{s,t}}, \quad \forall s, t \in S,\ m_{s,t} \neq \infty \,\right\rangle$$

encoded by a graph (the Coxeter graph) with edges labeled by $m_{s,t}$:

Convention: $m_{s,t} = 2$: no edge, $m_{s,t} = 3$: label omitted

Examples:

braid group



free group F_3

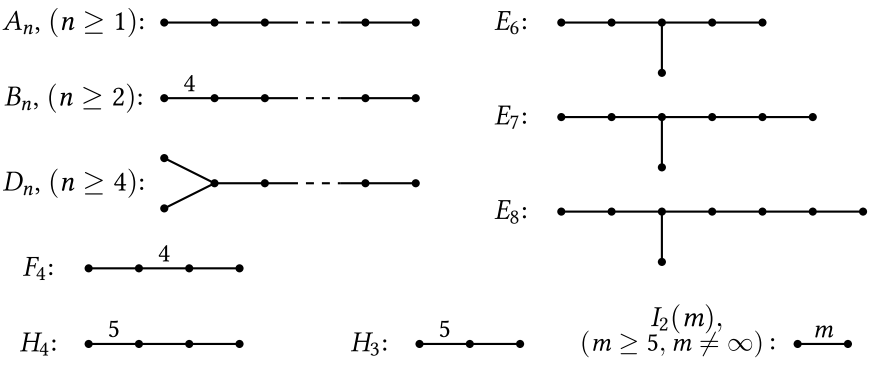
Natural questions:

- W word problem solvable?
- C conjugacy problem solvable?
- T torsion-free?
- Z center is trivial/cyclic?
- B biautomatic?

Classes of Artin groups for which we know the affirmative answers:

- **Spherical:** the Coxeter group $W = A/\langle s^2 \rangle$ is finite — W, C, T, Z, B
- **Right-angled:** all $m_{s,t}$ are either 2 or ∞ — W, C, T, Z, B
- **Extra large:** all $m_{s,t} \geq 4$ — W, C, T, B
- **Affine/euclidean:** given by ‘extended Dynkin diagrams’ — W, T, Z
- **FC-type**: non-infinite labeled edges span a spherical group — W

Spherical Artin groups (Coxeter (1935), Paris (2004))



Q: Which spherical Artin groups are quasi-isometric? (mostly) open

Q: Which spherical Artin groups are commensurable?

Partial answers by: **Cumplido–Paris (2021),** **S. (2021)**

Recall that two groups are **commensurable** ($\approx$) if they have isomorphic subgroups of finite index. (commensurable $\Rightarrow$ quasi-isometric)

Theorem (Cumplido–Paris (2021))

1. *If two spherical Artin groups are commensurable, then their ranks are equal and the corresponding irreducible components are commensurable to each other.*
2. *The only irreducible spherical Artin groups commensurable with that of type A_n are:*

$$A(A_n) \approx A(B_n) \quad \textit{and} \quad A(A_2) \approx A(I_2(m)).$$

This leaves six cases to be determined:

$$(F_4, D_4), \quad (H_4, D_4), \quad (F_4, H_4), \quad (D_6, E_6), \quad (D_7, E_7), \quad (D_8, E_8).$$

Theorem (S. (2021))

$$A(F_4) \not\approx A(D_4) \quad \textit{and} \quad A(H_4) \not\approx A(D_4).$$

Methods of Cumplido and Paris

Commensurability: $A(B_n) \leq_{f.i.} A(A_n)$ and $F_2 \leq_{f.i.} I_2(m)$, for all m.

Non-commensurability: Need to prove that there is no embedding

$$\overline{A(\Gamma)} \longrightarrow \mathrm{Comm}(\overline{A(A_n)})$$

where $\overline{A(\Gamma)} = A(\Gamma)/\text{center}$, and $\mathrm{Comm}(G)$ is the abstract commensurator.

$$\mathrm{Comm}(\overline{A(A_n)}) \cong \begin{pmatrix}\text{extended m.c.g. of sphere} \\ \text{with } n+2 \text{ punctures}\end{pmatrix} \cong \overline{A(A_n)} \rtimes \{\pm 1\}$$

They classify with the help of GAP all homomorphisms

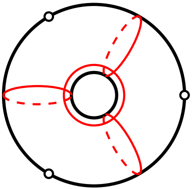
$$\overline{A(\Gamma)} \longrightarrow \mathrm{Comm}(\overline{A(A_n)}) \longrightarrow \mathrm{Comm}(\overline{A(A_n)})/\text{pure braids} = \mathfrak{S}_{n+2} \times \{\pm 1\}$$

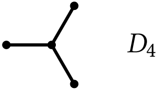
and produce generalized torsion in the bi-orderable kernel (pure braid group), which yields contradiction.

My approach:

Theorem (S. (2020,2021))

$$\mathrm{Comm}(\overline{A(D_4)}) \cong \left(\begin{matrix} \textit{extended m.c.g. of torus} \\ \textit{with 3 punctures} \end{matrix} \right) \cong \overline{A(D_4)} \rtimes (\mathfrak{S}_3 \times \{\pm 1\})$$





For nonexistence of $\overline{A(H_4)} \hookrightarrow \mathrm{Comm}(\overline{A(D_4)})$ I classified all torsion elements in groups $\overline{A(\Gamma)}$.

For nonexistence of $\overline{A(F_4)} \hookrightarrow \mathrm{Comm}(\overline{A(D_4)})$ I classified with the help of MAGMA all homomorphisms

$$\overline{A(\Gamma)} \longrightarrow \mathrm{Comm}(\overline{A(D_4)})/\text{bi-orderable f.i. subgroup} = \text{group of order } 1156.$$

Open problems:

P1: Establish non/commensurability for pairs (D_6, E_6), (D_7, E_7), (D_8, E_8).
(work in progress)
P2*: Establish non/commensurability for the pair (F_4, H_4).
P3*: Problem of quasi-isometry classification of spherical Artin groups.

References

- María Cumplido, Luis Paris, Commensurability in Artin groups of spherical type. ***Revista Matematica Iberoamericana*** 2021, in press.
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Thank you!